Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t = -1$

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November 19, 2012

Abstract

We prove that the hyperelliptic Torelli group is generated by Dehn twists about separating curves that are preserved by the hyperelliptic involution. This verifies a conjecture of Hain. The hyperelliptic Torelli group can be identified with the kernel of the Burau representation evaluated at $t = -1$ and also the fundamental group of the branch locus of the period mapping, and so we obtain analogous generating sets for those. One application is that each component in Torelli space of the locus of hyperelliptic curves becomes simply connected when curves of compact type are added.

1 Introduction

In this paper, we find simple generating sets for three closely related groups.

1. The hyperelliptic Torelli group $\mathcal{SI}_g$, that is, the subgroup of the mapping class group consisting of elements that commute with the hyperelliptic involution and that act trivially on the homology of the surface.
2. The fundamental group of $\mathcal{H}_g$, the branch locus of the period mapping from Torelli space to the Siegel upper half plane.
3. The kernel of β_n, the Burau representation of the braid group evaluated at $t = -1$.

The group $\mathcal{SI}_g$, the space $\mathcal{H}_g$, and the representation β_n arise in many places in algebraic geometry, number theory, and topology. See, for instance, work of A'Campo [1], Arnol'd [2], Funar–Kohno [18], Hain [19], Khovanov–Seidel [24], Magnus–Peluso [26], McMullen [30], Morifuji [32], and Yu [40].

Hyperelliptic Torelli group. Let Σ_g be a closed oriented genus g surface and let Mod_g be its mapping class group, that is, the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g.

Let $\iota : \Sigma_g \to \Sigma_g$ be a hyperelliptic involution; see Figure 1. The hyperelliptic mapping class group SMod_g is the subgroup of Mod_g consisting of mapping classes that can be represented by homeomorphisms that commute with ι.

The Torelli group $\mathcal{I}_g$ is the kernel of the action of Mod_g on $H_1(\Sigma_g; \mathbb{Z})$, and the hyperelliptic Torelli group $\mathcal{SI}_g$ is $\mathrm{SMod}_g \cap \mathcal{I}_g$.

*The first author is supported in part by the EPSRC. The second author is supported in part by a Sloan Fellowship and an NSF CAREER award. The third author is supported in part by NSF grant DMS-1005318.

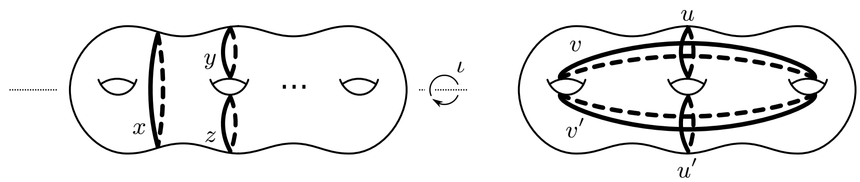

Figure 1: *The hyperelliptic involution ι rotates the surface 180 degrees about the indicated axis. The mapping class T_x is a Dehn twist about a symmetric separating curve. A bounding pair map, such as $T_yT_z^{-1}$, is the difference of two Dehn twists about disjoint, nonseparating, homologous simple closed curves. The elements $T_uT_{u'}$ and $T_vT_{v'}$ are in SMod_g and their actions on $H_1(\Sigma_g;\mathbb{Z})$ commute because $\hat{\imath}(u,v) = \hat{\imath}(u',v') = 0$, so $[T_uT_{u'}, T_vT_{v'}] \in \mathcal{SI}_g$.*

A simple closed curve x in Σ_g is *symmetric* if $\iota(x) = x$, in which case the Dehn twist T_x is in SMod_g. If x is a separating curve, then $T_x \in \mathcal{I}_g$; see Figure 1.

Theorem A. *For $g \geq 0$, the group $\mathcal{SI}_g$ is generated by Dehn twists about symmetric separating curves.*

The first two authors proved that Theorem A in fact implies the stronger result that $\mathcal{SI}_g$ is generated by Dehn twists about symmetric separating curves that cut off subsurfaces of genus 1 and 2; see [9, Section 5].

Theorem A was conjectured by Hain [19, Conjecture 1] and is also listed as a folk conjecture by Morifuji [32, Section 4]. Hain has informed us that he has proven the case $g = 3$ of Theorem A. His proof uses special properties of the Schottky locus in genus 3.

When we first encountered Hain's conjecture, it appeared to us to be overly optimistic. There is a well-known generating set for $\mathcal{I}_g$, namely, the set of bounding pair maps and Dehn twists about separating curves; see Figure 1. There is no reason to expect that a subgroup of $\mathcal{I}_g$ should be generated by the elements on this list lying in the subgroup. Additionally, there are several other natural elements of $\mathcal{SI}_g$, and it was not at first clear how to write those in terms of Hain's proposed generators. Consider for instance the mapping class $[T_uT_{u'}, T_vT_{v'}] \in \mathcal{SI}_g$ indicated in Figure 1. Eventually, it turned out this element is a product of six Dehn twists about symmetric separating curves [10], but those symmetric separating curves are rather complicated looking.

Branch locus of the period map. Hain [19] observed that Theorem A has another interpretation in terms of the period map. Let $\mathcal{T}_g$ be Teichmüller space and $\mathfrak{h}_g$ be the Siegel upper half plane. The period map $\mathcal{T}_g \to \mathfrak{h}_g$ takes a Riemann surface to its Jacobian. It factors through the Torelli space $\mathcal{T}_g/\mathcal{I}_g$, which is an Eilenberg–MacLane space for $\mathcal{I}_g$. The induced map $\mathcal{T}_g/\mathcal{I}_g \to \mathfrak{h}_g$ is a 2-fold branched cover onto its image. The branch locus is the subspace $\mathcal{H}_g \subset \mathcal{T}_g/\mathcal{I}_g$ consisting of hyperelliptic curves. The space $\mathcal{H}_g$ is not connected, but its components are all homeomorphic and have fundamental group $\mathcal{SI}_g$. Thus, Theorem A gives generators for $\pi_1(\mathcal{H}_g)$.

Let $\mathcal{H}_g^c$ be the space obtained by adjoining hyperelliptic curves of compact type to $\mathcal{H}_g$. Theorem A has the following corollary.

Theorem B. *For $g \geq 0$, each component of $\mathcal{H}_g^c$ is simply connected.*

See Hain's paper [19] for the details on how to derive Theorem B from Theorem A.

Kernel of the Burau representation. The Burau representation [5] is an important representation of the braid group B_n to $\mathrm{GL}_{n-1}(\mathbb{Z}[t,t^{-1}])$. Let $\beta_n : \mathrm{B}_n \to \mathrm{GL}_{n-1}(\mathbb{Z})$ denote the representation obtained by substituting $t=-1$ into the (reduced) Burau representation. Denote the kernel of β_n by $\mathcal{BI}_n$ (the notation stands for "braid Torelli group").

We identify B_n with the mapping class group of a disk D_n with n marked points, that is, the group of isotopy classes of homeomorphisms of D_n preserving the set of marked points and fixing $\partial\,\mathrm{D}_n$ pointwise. For most purposes, we will regard the marked points as punctures. For instance curves (and homotopies of curves) are not allowed to pass through the marked points. When we say that a simple closed curve in essential in D_n we mean that is it not homotopic to a marked point, an unmarked point, or the boundary.

Theorem C. *For $n \geq 1$, the group $\mathcal{BI}_n$ is generated by squares of Dehn twists about curves in D_n surrounding odd numbers of marked points.*

Hyperelliptic Torelli vs Burau. We now explain the relationship between Theorems A and C. This requires defining the hyperelliptic Torelli group for a surface with boundary.

Let Σ_g^1 be the surface obtained from Σ_g by deleting the interior of an embedded ι-invariant disk. There is an induced hyperelliptic involution of Σ_g^1 which we also call ι. Let Mod_g^1 be the group of isotopy classes of homeomorphisms of Σ_g^1 that fix $\partial\Sigma_g^1$ pointwise and let SMod_g^1 be the subgroup of Mod_g^1 consisting of mapping classes that can be represented by homeomorphisms that commute with ι. Observe that unlike for Mod_g, the map ι does not correspond to an element of Mod_g^1. Finally, let $\mathcal{I}_g^1$ be the kernel of the action of Mod_g^1 on $H_1(\Sigma_g^1;\mathbb{Z})$ and let $\mathcal{SI}_g^1 = \mathrm{SMod}_g^1 \cap \mathcal{I}_g^1$.

The involution ι fixes $2g+1$ points on Σ_g^1. Regarding the images of these points in Σ_g^1/ι as marked points, we have $\Sigma_g^1/\iota \cong \mathrm{D}_{2g+1}$. There is a homomorphism $L : \mathrm{B}_{2g+1} \to \mathrm{SMod}_g^1$ which lifts a mapping class through the branched cover $\Sigma_g^1 \to \Sigma_g^1/\iota$. Birman–Hilden [6] proved that L is an isomorphism.

The representation β_{2g+1} is conjugate to the composition

$$\mathrm{B}_{2g+1} \xrightarrow{L} \mathrm{SMod}_g^1 \hookrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}),$$

where the map $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is the standard representation arising from the action of Mod_g^1 on $H_1(\Sigma_g^1;\mathbb{Z})$. The map L therefore restricts to an isomorphism $\mathcal{BI}_{2g+1} \cong \mathcal{SI}_g^1$. Under this isomorphism, squares of Dehn twists about curves surrounding odd numbers of marked points map to Dehn twists about symmetric separating curves. For a fixed $g \geq 1$, the case $n = 2g+1$ of Theorem C is therefore equivalent to the statement that $\mathcal{SI}_g^1$ is generated by Dehn twists about symmetric separating curves. The first two authors proved [9] that the kernel of the natural map $\mathcal{SI}_g^1 \to \mathcal{SI}_g$ is generated by the Dehn twist about $\partial\Sigma_g^1$, so this is equivalent to Theorem A.

We can also relate Theorem C for even numbers of punctures to the mapping class group by extending Theorem A to the case of a surface with two boundary components. Briefly, let Σ_g^2 be the compact surface of genus g with two boundary components obtained by removing the interiors of two disks in Σ_g that are interchanged by ι. Again, there is an induced hyperelliptic involution of Σ_g^2 which we will also call ι. The homeomorphism ι interchanges

the two boundary components of Σ_g^2. We can define SMod_g^2 as before. The Torelli group $\mathcal{I}_g^2$ is the kernel of the action of SMod_g^2 on $H_1(\Sigma_g^2, P; \mathbb{Z})$, where P is a pair of points, one on each boundary component of Σ_g^2. The hyperelliptic Torelli group $\mathcal{SI}_g^2 = \mathrm{SMod}_g^2 \cap \mathcal{I}_g^2$ is then isomorphic to $\mathcal{BI}_{2g+2}$. The $n = 2g+2$ case of Theorem C translates to the fact that $\mathcal{SI}_g^2$ is generated by Dehn twists about symmetric separating curves. See [9] for more details.

Prior results. Theorem A was previously known for $g \leq 2$. It is a classical fact that $\mathcal{I}_g = 1$ for $g \leq 1$, so $\mathcal{SI}_g$ is trivial in these cases. When $g = 2$, all essential curves in Σ_g are homotopic to symmetric curves. This implies that $\mathrm{SMod}_g = \mathrm{Mod}_g$ and $\mathcal{SI}_g = \mathcal{I}_g$ (see, e.g., [17, Section 9.4.2]). The group $\mathcal{I}_2$ is generated by Dehn twists about separating curves; in fact, Mess [31] proved that $\mathcal{I}_2$ is a free group on an infinite set of Dehn twists about separating curves (McCullough–Miller [29] previously showed $\mathcal{I}_2$ was infinitely generated). This implies that $\mathcal{SI}_2$ is generated by Dehn twists about symmetric separating curves.

Aside from our Theorem A, little is known about $\mathcal{SI}_g$ when $g \geq 3$. Letting $H = H_1(\Sigma_g; \mathbb{Z})$, Johnson [22, 23] constructed a Mod_g-equivariant homomorphism

$$\tau : \mathcal{I}_g \to (\wedge^3 H)/H$$

and proved that $\ker(\tau)$ is precisely the subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$ generated by Dehn twists about (not-necessarily-symmetric) separating curves. Since ι acts by $-\mathrm{Id}$ on $(\wedge^3 H)/H$, it follows that $\mathcal{SI}_g < \mathcal{K}_g$. Despite the fact that $\mathcal{SI}_g$ has infinite index in $\mathcal{K}_g$, Childers [15] showed that these groups have the same image in the abelianization of $\mathcal{I}_g$.

Work of Birman [4] and Powell [33] shows that $\mathcal{I}_g$ is generated by bounding pair maps and Dehn twists about separating curves; other proofs were given by Putman [34] and by Hatcher–Margalit [20]. One can find bounding pair maps $T_y T_z^{-1}$ such that ι exchanges y and z (see Figure 1); however, these do not lie in $\mathcal{SI}_g$ since $\iota T_y T_z^{-1} \iota^{-1} = T_z T_y^{-1}$. In fact, since no power of a bounding pair map is in $\ker(\tau)$, there are no nontrivial powers of bounding pair maps in $\mathcal{SI}_g$.

With Childers, the first two authors proved that $\mathcal{SI}_g$ has cohomological dimension $g-1$ and that $H_{g-1}(\mathcal{SI}_g; \mathbb{Z})$ has infinite rank [12]. This implies that $\mathcal{SI}_3$ is not finitely presentable. It is not known, however, whether $\mathcal{SI}_g$, or even $H_1(\mathcal{SI}_g; \mathbb{Z})$, is finitely generated for $g \geq 3$.

Approach of the paper. One's first impulse is to try to generalize to the setting of the hyperelliptic Torelli group one of the known proofs that the mapping class group is generated by Dehn twists or that the Torelli group is generated by separating twists and bounding pair maps. Unfortunately, any naïve translation of these proofs to the case of $\mathcal{SI}_g$ immediately runs into serious problems, and so our proof requires a new approach.

First, it suffices to prove Theorem C for $n = 2g+1$. Indeed, the first two authors proved [9] that the $n = 2g+1$ case of Theorem C is equivalent to the genus g case of Theorem A and that the $n = 2g+1$ case of Theorem C implies the $n = 2g+2$ case of Theorem C.

It follows from work of Arnol'd and A'Campo (see Section 2) that $\ker(\beta_{2g+1})$ is contained in the pure braid group PB_{2g+1} and that the image of PB_{2g+1} under β_{2g+1} is the level 2 subgroup $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Letting $\Theta_{2g+1} \subset \mathrm{PB}_{2g+1}$ be the group generated by squares of Dehn twists about curves surrounding odd numbers of marked points, Theorem C is equivalent to the assertion that $\mathrm{PB}_{2g+1} / \Theta_{2g+1} \cong \mathrm{Sp}_{2g}(\mathbb{Z})[2]$.

This isomorphism can be viewed as giving a finite presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ since PB_{2g+1} is finitely presented and Θ_{2g+1} has a finite normal generating set. There are several known presentations for $\mathrm{Sp}_{2g}(\mathbb{Z})$. Also, there are standard tools for obtaining presentations of finite-index subgroups of finitely presented groups (e.g. Reidemeister–Schreier). However, they all explode in complexity as the index of the subgroup grows. The index of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is exponential in g, so it is not feasible to use these tools here. We instead introduce new tools for obtaining presentations of normal subgroups of groups acting on simplicial complexes. We apply these tools to $\mathrm{Sp}_{2g}(\mathbb{Z})[2] \lhd \mathrm{Sp}_{2g}(\mathbb{Z})$, which act on simplicial complexes we construct which are related to buildings.

The tools we construct should be useful in other contexts. They have already been used by the last two authors to give finite presentations for certain congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ which are reminiscent of the standard presentation for $\mathrm{SL}_n(\mathbb{Z})$; see [28].

Acknowledgments. We would like to thank Joan Birman, Benson Farb, Richard Hain, and Curt McMullen for helpful conversations.

2 Outline of paper

We now describe in more detail the proof of Theorem C for odd n and at the same time give the plan for the rest of the paper. Since $\mathcal{BI}_{2g+1} = \Theta_{2g+1}$ for $0 \leq g \leq 2$, we can assume by induction that $g \geq 3$ and that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$, and then prove that $\mathcal{BI}_{2g+1} = \Theta_{2g+1}$. This is the content of Proposition 2.1 below. After its statement, we give an outline of the proof and a plan for the remainder of the paper.

Background. Arnol'd [2] proved that the kernel of the composition

$$\mathrm{B}_{2g+1} \xrightarrow{\beta_{2g+1}} \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/2)$$

is exactly PB_{2g+1}. In particular, the image of PB_{2g+1} under β_{2g+1} lies in the level 2 congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$, namely $\mathrm{Sp}_{2g}(\mathbb{Z})[2] = \ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/2))$. A'Campo [1] then proved that $\beta_{2g+1}(\mathrm{PB}_{2g+1})$ is all of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$. These two results can be summarized in the following commutative diagram. In the diagram, S_{2g+1} is the symmetric group, and the map $\mathrm{B}_{2g+1} \to S_{2g+1}$ is the action on the marked points of D_{2g+1}.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{PB}_{2g+1} \arrow[r] \arrow[d, two heads] & \mathrm{B}_{2g+1} \arrow[r] \arrow[d, "\beta_{2g+1}"'] & S_{2g+1} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Sp}_{2g}(\mathbb{Z})[2] \arrow[r] & \mathrm{Sp}_{2g}(\mathbb{Z}) \arrow[r] & \mathrm{Sp}_{2g}(\mathbb{Z}/2) \arrow[r] & 1
\end{tikzcd}$$

In particular, we see that $\mathcal{BI}_{2g+1} < \mathrm{PB}_{2g+1}$ and $\mathrm{PB}_{2g+1} / \mathcal{BI}_{2g+1} \cong \mathrm{Sp}_{2g}(\mathbb{Z})[2]$.

The Main Proposition. Denote the quotient $\mathrm{PB}_{2g+1} / \Theta_{2g+1}$ by $\mathcal{Q}_g$. The $n = 2g+1$ case of Theorem C can be stated as $\mathcal{BI}_{2g+1} = \Theta_{2g+1}$. By the above discussion, this is equivalent to the following.

Proposition 2.1. *Suppose that $g \geq 3$ and that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for $h < g$. Then the quotient map $\mathcal{Q}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is an isomorphism.*

Proposition 2.1 is proven in Section 3 modulo two technical statements, namely, Propositions 3.3 and 3.4 below.

Outline of the proof of the Main Proposition. To prove Proposition 2.1, it suffices to construct an inverse map $\phi : \mathrm{Sp}_{2g}(\mathbb{Z})[2] \to \mathcal{Q}_g$. There are two main ingredients to the construction. We describe them and at the same time give an outline for the rest of the paper.

1. The first ingredient is an infinite presentation for the group $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$. The third author developed a method for constructing infinite presentations of groups using group actions on complexes and used this to construct an infinite presentation of the Torelli group [35]. The general construction is recalled in Section 3.1.1. The complexes $\mathfrak{IB}_g(\mathbb{Z})$ we use are discussed in Section 3.1.2; they are related to buildings. The theory requires $\mathfrak{IB}_g(\mathbb{Z})$ and its quotient to have certain connectivity properties, which are given in Proposition 3.3 and proven in Section 5.

2. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ by conjugation. Our second ingredient, which is discussed in Section 3.2 below, is an analogous action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$; see Proposition 3.4. We construct the action by declaring where each generator of $\mathrm{Sp}_{2g}(\mathbb{Z})$ sends each generator of $\mathcal{Q}_g$ and then checking that all relations in both groups are satisfied. This step uses a mixture of surface topology and combinatorial group theory, and the proofs are relegated to Section 6.

Our infinite presentation of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ will be compatible in a certain technical sense with the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ by conjugation; see Proposition 3.4(1). In Section 4 we will use this compatibility together with our action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ to map each piece of our presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ to $\mathcal{Q}_g$, thus proving Proposition 2.1.

Factoring reducible elements. At various points in the proof, specifically Sections 4, 6.4, and 6.5, we will appeal to the following theorem of the first two authors [9], which which makes it easy to recognize elements of Θ_{2g+1}. In the statement, an element of $\mathcal{BI}_{2g+1}$ is *reducible* if it fixes the homotopy class of an essential simple closed curve.

Theorem 2.2. *If $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$, then all reducible elements of $\mathcal{BI}_{2g+1}$ lie in Θ_{2g+1}.*

This theorem is derived from a version of the Birman exact sequence for $\mathcal{SI}_g$.

3 Main Ingredients

As in Section 2, there are two main ingredients to the proof of Proposition 2.1, namely, the infinite presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ and the group action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$. In this section we discuss each of these in turn.

3.1 An infinite presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$

In this section we give the third author's theorem about obtaining presentations from group actions and then we discuss the specific simplicial complexes and group actions to which this theorem will be applied.

3.1.1 A presentation theorem

Let G be a group acting without rotations on a simplicial complex X; this means that if an element of G preserves some simplex of X then it fixes that simplex pointwise. For a simplex σ, write G_σ for the stabilizer of σ. Also write $X^{(0)}$ for the vertex set of X. There is a homomorphism

$$\psi : \mathop{\ast}_{v \in X^{(0)}} G_v \longrightarrow G.$$

If $a \in G$ stabilizes $v \in X^{(0)}$, then we denote a considered as an element of

$$G_v < \mathop{\ast}_{v \in X^{(0)}} G_v$$

by a_v. There are some obvious elements in $\ker(\psi)$. First, we have $a_v a_{v'}^{-1} \in \ker(\psi)$ if v and v' are joined by an edge e and $a \in G_e$. We call these the *edge relators*. Second, we have $b_w a_v b_w^{-1} (bab^{-1})_{b(v)}^{-1} \in \ker(\psi)$ for $a \in G_v$ and $b \in G_w$. We call these the *conjugation relators*. The third author proved [36] that under favorable circumstances these normally generate $\ker(\psi)$.

Theorem 3.1. *Let G be a group acting without rotations on a simplicial complex X. Assume that X is 1-connected and X/G is 2-connected. Then*

$$G \cong \left(\mathop{\ast}_{v \in X^{(0)}} G_v \right) \Big/ R,$$

where R is the normal closure of the edge and conjugation relators.

3.1.2 Building-like simplicial complexes

We will apply Theorem 3.1 to the action of the group $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ on the complex of isotropic bases $\mathfrak{IB}_g(\mathbb{Z})$, which we presently define. This complex will be simply connected for $g \geq 3$, as demanded by Theorem 3.1. However, the quotient $\mathfrak{IB}_g(\mathbb{Z})/\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is only $(g-2)$-connected. Therefore, for $g = 3$, we will need to modify the complex in order to attain 2-connectivity. The modified complex will be called the augmented complex of partial bases $\widehat{\mathfrak{IB}}_g(R)$, and is defined below.

Complex of isotropic bases. Let R be either $\mathbb{Z}$ or a field, and let $\hat{\imath}$ be the standard symplectic form on R^{2g}. A subspace V of R^{2g} is a *free summand* if V is a free R-module and $R^{2g} = V \oplus W$ for some submodule W of R^{2g}. It is *isotropic* if $\hat{\imath}(\vec{v}, \vec{w}) = 0$ for all $\vec{v}, \vec{w} \in V$. The first version of our complex is then as follows.

The *complex of isotropic bases* for R^{2g}, denoted $\mathfrak{IB}_g(R)$, is the simplicial complex whose k-simplices are the sets $\{\vec{v}_0, \ldots, \vec{v}_k\}$ of vectors in R^{2g} that form a basis for an isotropic free summand of R^{2g}.

Trouble in low genus. As recorded in Proposition 3.2(1) below, the group $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ acts without rotations on $\mathfrak{IB}_g(\mathbb{Z})$, and the quotient is isomorphic to $\mathfrak{IB}_g(\mathbb{Z}/2)$. And by Proposition 3.3(1) below, the complexes $\mathfrak{IB}_g(\mathbb{Z})$ and $\mathfrak{IB}_g(\mathbb{Z}/2)$ are both $(g-2)$-connected. Therefore, the action of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ on $\mathfrak{IB}_g(\mathbb{Z})$ satisfies the hypotheses of Theorem 3.1, but only for $g \geq 4$. We still have the genus 3 case to contend with, and we address this issue by adding cells to $\mathfrak{IB}_g(\mathbb{Z})$, as follows.

Augmented complex of isotropic bases. Let R and $\hat{\imath}$ be as above. A k-simplex $\{\vec{v}_0, \ldots, \vec{v}_k\}$ of $\mathfrak{IB}_g(R)$ will be called a *standard k-simplex.* Suppose $\{\vec{v}_0, \ldots, \vec{v}_k\}$ is a standard k-simplex. A simplex $\{\vec{w}, \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_k\}$ is a *simplex of additive type* if either $k \geq 1$ and $w = \vec{v}_0 \pm \vec{v}_1$ or $k \geq 2$ and $w = \vec{v}_0 \pm \vec{v}_1 \pm \vec{v}_2$. Also, $\{\vec{w}, \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_k\}$ is a *simplex of intersection type* if $\vec{w} \in R^{2g}$ satisfies $\hat{\imath}(\vec{v}_0, \vec{w}) = 1$ and $\hat{\imath}(\vec{v}_i, \vec{w}) = 0$ for $i \geq 1$. An equivalent condition is that $\hat{\imath}(\vec{v}_0, \vec{w}) = 1$ and both $\{\vec{v}_0, \ldots, \vec{v}_k\}$ and $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ are standard simplices. The *augmented complex of isotropic bases* $\widehat{\mathfrak{IB}}_g(R)$ is the simplicial complex whose simplices are the standard simplices and the simplices of additive and intersection type.

Group action and connectivity. Clearly $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on both $\mathfrak{IB}_g(\mathbb{Z})$ and $\widehat{\mathfrak{IB}}_g(\mathbb{Z})$. The following describes the restriction of this action to $\mathrm{Sp}_{2g}(\mathbb{Z})[p]$.

Proposition 3.2. *Let $p \geq 2$ be prime.*

1. *The group $\mathrm{Sp}_{2g}(\mathbb{Z})[p]$ acts without rotations on $\mathfrak{IB}_g(\mathbb{Z})$ and*
$$\mathfrak{IB}_g(\mathbb{Z}) / \mathrm{Sp}_{2g}(\mathbb{Z})[p] \cong \mathfrak{IB}_g(\mathbb{Z}/p).$$
2. *The group $\mathrm{Sp}_{2g}(\mathbb{Z})[p]$ acts without rotations on $\widehat{\mathfrak{IB}}_g(\mathbb{Z})$ and*
$$\widehat{\mathfrak{IB}}_g(\mathbb{Z}) / \mathrm{Sp}_{2g}(\mathbb{Z})[p] \cong \widehat{\mathfrak{IB}}_g(\mathbb{Z}/p).$$

Proposition 3.2 can be proved exactly like [37, Proposition 6.7].

Our main result about our complexes of isotropic bases is the following.

Proposition 3.3. *Let R be either $\mathbb{Z}$ or a field.*

1. *The complex $\mathfrak{IB}_g(R)$ is homotopy equivalent to a wedge of $(g-1)$-spheres.*
2. *The complexes $\widehat{\mathfrak{IB}}_3(\mathbb{Z})$ and $\widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ are 1-connected and 2-connected, respectively.*

The proof of Proposition 3.3(1) is in Section 5.2. It is similar to a proof of a related result due to Charney; see [14, Theorem 2.9]. Charney's result applies to more general rings but has a weaker range of connectivity. Proposition 3.3(2) is proven in Section 5.3.

3.2 A symplectic group action

We now discuss the second ingredient for the proof of Proposition 2.1, namely, the group action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ by conjugation. We wish

to lift this to an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ in a natural way. Let

$$\mathcal{Q}_g = \mathrm{PB}_{2g+1}/\Theta_{2g+1} \quad \text{and} \quad \widehat{\mathcal{Q}}_g = \mathrm{B}_{2g+1}/\Theta_{2g+1},$$

and then let

$$\rho : \mathrm{PB}_{2g+1} \to \mathcal{Q}_g, \qquad \widehat{\rho} : \mathrm{B}_{2g+1} \to \widehat{\mathcal{Q}}_g,$$
$$\pi : \mathcal{Q}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})[2], \quad \text{and} \quad \widehat{\pi} : \widehat{\mathcal{Q}}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$$

be the quotient maps. The first two parts of Proposition 3.4 below posit the existence of an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ that is natural with respect to the actions of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ and of $\widehat{\mathcal{Q}}_g$ on $\mathcal{Q}_g$.

We will require our action to have one extra property, which requires some setup. Let c_{23} be the curve in Σ_g^1/ι shown in Figure 2 and let $\vec{v}_{23}$ be the element of $H_1(\Sigma_g^1;\mathbb{Z})$ represented by one component of the preimage of c_{23} in Σ_g^1 with some choice of orientation. Denote by $(\mathrm{PB}_{2g+1})_{c_{23}}$ the stabilizer in PB_{2g+1} of the isotopy class of c_{23} and by $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$ the stabilizer in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of $\vec{v}_{23}$. Define

$$\Omega_{23} = \rho\left((\mathrm{PB}_{2g+1})_{c_{23}}\right) \subseteq \mathcal{Q}_g.$$

It is clear that the action of $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$ on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ preserves $\pi(\Omega_{23})$.

Proposition 3.4. *Let $g \geq 3$. Assume that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$. There then exists an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on the group $\mathcal{Q}_g$ with the following three properties:*

1. *For $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\eta \in \mathcal{Q}_g$, we have $\pi(Z \cdot \eta) = Z\pi(\eta)Z^{-1}$.*
2. *For $\nu \in \widehat{\mathcal{Q}}_g$ and $\eta \in \mathcal{Q}_g$, we have $\widehat{\pi}(\nu) \cdot \eta = \nu\eta\nu^{-1}$.*
3. *The action of $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$ on $\mathcal{Q}_g$ preserves Ω_{23}.*

Proposition 3.4 is proved in Section 6.

4 The proof of the Main Proposition

In this section we prove Proposition 2.1 assuming Propositions 3.3 and 3.4. Recall that we are assuming $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$ and want to prove that the quotient map $\pi : \mathcal{Q}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is an isomorphism. The map π is a surjection, so it is enough to construct a homomorphism $\phi : \mathrm{Sp}_{2g}(\mathbb{Z})[2] \to \mathcal{Q}_g$ such that $\phi \circ \pi = 1$.

Let $X_g = \mathfrak{IB}_g(\mathbb{Z})$ if $g \geq 4$ and $X_g = \widehat{\mathfrak{IB}}_g(\mathbb{Z})$ if $g = 3$. Propositions 3.2 and 3.3 show that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ on X_g satisfies the conditions of Theorem 3.1 if $g \geq 3$. Theorem 3.1 then gives

$$\mathrm{Sp}_{2g}(\mathbb{Z})[2] \cong \left(\mathop{\ast}_{\vec{v} \in X_g^{(0)}} \left(\mathrm{Sp}_{2g}(\mathbb{Z})[2]\right)_{\vec{v}} \right) \Big/ R,$$

where R is the normal closure of the edge and conjugation relators.

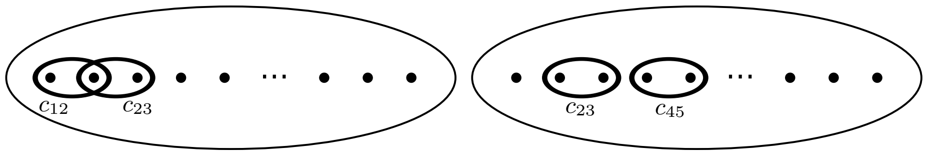

Figure 2: *The curves $\{c_{12}, c_{23}\}$ in Σ_g^1/ι lift to loops in Σ_g^1 whose homology classes form an edge e_1 of intersection type in $\widehat{\mathfrak{IB}}_g$. The curves $\{c_{23}, c_{45}\}$ in Σ_g^1/ι lift to loops whose homology classes form a standard edge e_2 in $\widehat{\mathfrak{IB}}_g$.*

We will construct the map ϕ in two steps. First we will use Proposition 3.4 to construct a homomorphism

$$\widetilde{\phi} : \mathop{\ast}_{\vec{v} \in X_g^{(0)}} \left(\mathrm{Sp}_{2g}(\mathbb{Z})[2]\right)_{\vec{v}} \to \mathcal{Q}_g$$

(recall that Proposition 3.4 requires the assumption that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$). Then we will show that $\widetilde{\phi}$ takes the edge and conjugation relators to the identity (Lemmas 4.4 and 4.5), so it induces a homomorphism $\phi : \mathrm{Sp}_{2g}(\mathbb{Z})[2] \to \mathcal{Q}_g$. Finally we will check that $\phi \circ \pi$ is equal to the identity (Lemma 4.6), completing the proof.

In addition to π and $\mathcal{Q}_g$, we will use the notations $\widehat{\mathcal{Q}}_g$, ρ, $\widehat{\rho}$, $\widehat{\pi}$, c_{23}, $\vec{v}_{23}$, and Ω_{23} from Section 3.2.

Construction of $\widetilde{\phi}$. For each $\vec{v} \in X_g^{(0)}$, we need to construct a homomorphism

$$\widetilde{\phi}_{\vec{v}} : (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}} \to \mathcal{Q}_g.$$

We start by dealing with the special case $\vec{v} = \vec{v}_{23}$. By construction, $\ker(\pi|_{\Omega_{23}})$ lies in the image in $\mathcal{Q}_g$ of the group $(\mathrm{PB}_{2g+1})_{c_{23}}$. Recall that we are assuming that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for all $h < g$, so Theorem 2.2 implies that reducible elements of $\mathcal{BI}_{2g+1}$ lie in Θ_{2g+1} (and hence map to 1 in $\mathcal{Q}_g$). We conclude that $\ker(\pi|_{\Omega_{23}}) = 1$, i.e. that $\pi|_{\Omega_{23}}$ is injective. The first two authors proved [11] that $\pi(\Omega_{23})$ is in fact equal to $(\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{v_{23}}$. We define $\widetilde{\phi}_{\vec{v}_{23}} = \pi|_{\Omega_{23}}^{-1}$.

We now consider a general $\vec{v} \in X_g^{(0)}$. Here we use the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ provided by Proposition 3.4. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on the vertices of X_g, so there exists some $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Z(\vec{v}_{23}) = \vec{v}$. We then define

$$\widetilde{\phi}_{\vec{v}}(Y) = Z \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ) \qquad \left(Y \in \left(\mathrm{Sp}_{2g}(\mathbb{Z})[2]\right)_{\vec{v}}\right).$$

Clearly $\widetilde{\phi}_{\vec{v}}$ is a homomorphism. It appears to depend on the choice of Z; however, this dependence is illusory.

Lemma 4.1. *Each map $\widetilde{\phi}_{\vec{v}}$ is a well-defined homomorphism.*

Proof. Say $Z_1, Z_2 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ both satisfy $Z_i(\vec{v}_{23}) = \vec{v}$. In this case $W = Z_1^{-1}Z_2$ lies in $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$. Let $U = Z_2^{-1}YZ_2$. We first claim that

$$\widetilde{\phi}_{\vec{v}_{23}}(WUW^{-1}) = W \cdot \widetilde{\phi}_{\vec{v}_{23}}(U)$$

To prove the claim, first notice that by the definition of $\widetilde{\phi}_{\vec{v}_{23}}$, both $\widetilde{\phi}_{\vec{v}_{23}}(WUW^{-1})$ and $\widetilde{\phi}_{\vec{v}_{23}}(U)$ lie in Ω_{23}. By Proposition 3.4(3), the element $W \cdot \widetilde{\phi}_{\vec{v}_{23}}(U)$ also lies in Ω_{23}. Since

$\pi|_{\Omega_{23}}$ is injective, it remains to show that $\widetilde{\phi}_{\vec{v}_{23}}(WUW^{-1})$ and $W \cdot \widetilde{\phi}_{\vec{v}_{23}}(U)$ have the same image under π. We have

$$\pi(\widetilde{\phi}_{\vec{v}_{23}}(WUW^{-1})) = WUW^{-1} = W\pi(\widetilde{\phi}_{\vec{v}_{23}}(U))W^{-1} = \pi(W \cdot \widetilde{\phi}_{\vec{v}_{23}}(U)),$$

where the first and second equalities use the fact that $\pi \circ \widetilde{\phi}_{\vec{v}_{23}}$ equals the identity and the third equality uses Proposition 3.4(1).

Using the claim, we now have

$$Z_1 \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z_1^{-1}YZ_1) = Z_1 \cdot \widetilde{\phi}_{\vec{v}_{23}}(WUW^{-1}) = Z_1 \cdot \Big(W \cdot \widetilde{\phi}_{\vec{v}_{23}}(U)\Big) = Z_2 \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z_2^{-1}YZ_2),$$

as desired. □

We will require the following two facts about the maps $\widetilde{\phi}_{\vec{v}}$.

Lemma 4.2. *For any $\vec{v} \in X_g^{(0)}$, we have $\pi \circ \widetilde{\phi}_{\vec{v}} = id$.*

Proof. Pick $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Z(\vec{v}_{23}) = \vec{v}$, and let $Y \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}}$. We have

$$\pi(\widetilde{\phi}_{\vec{v}}(Y)) = \pi(Z \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ)) = Z\pi(\widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ))Z^{-1} = ZZ^{-1}YZZ^{-1} = Y,$$

as desired. The first equality uses the definition of $\widetilde{\phi}_{\vec{v}}$, the second equality uses Proposition 3.4(2), and the third equality uses the fact that $\pi \circ \widetilde{\phi}_{\vec{v}_{23}}$ is equal to the identity. □

Lemma 4.3. *For any $\vec{v} \in X_g^{(0)}$, we have $\widetilde{\phi}_{\vec{v}} = \widetilde{\phi}_{-\vec{v}}$.*

Note that the equality $\widetilde{\phi}_{\vec{v}} = \widetilde{\phi}_{-\vec{v}}$ in Lemma 4.3 makes sense since the domains of these maps are equal, that is, $(\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}} = (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{-\vec{v}}$.

Proof of Lemma 4.3. Choose $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ so that $Z(\vec{v}_{23}) = \vec{v}$. We then have $(-Z)(\vec{v}_{23}) = -\vec{v}$. Let $C \in \widehat{\mathcal{Q}}_g$ be the image of the Dehn twist about ∂D_{2g+1}. This Dehn twist is central in B_{2g+1}, so C is central in $\widehat{\mathcal{Q}}_g$. It is easy to see that $\widehat{\pi}(C) = -I$. For $Y \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}}$, we have

$$\begin{aligned}\widetilde{\phi}_{\vec{v}}(Y) &= Z \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ) = C(Z \cdot \widetilde{\phi}_{\vec{v}_{23}}((-Z)^{-1}Y(-Z)))C^{-1} \\ &= (-I) \cdot \Big(Z \cdot \widetilde{\phi}_{\vec{v}_{23}}((-Z)^{-1}Y(-Z))\Big) = (-Z) \cdot \widetilde{\phi}_{\vec{v}_{23}}((-Z)^{-1}Y(-Z)) = \widetilde{\phi}_{-\vec{v}}(Y),\end{aligned}$$

as desired. The second equality uses the centrality of C and $-I$, the third equality uses Proposition 3.4(2), and the other equalities simply use the definition of $\widetilde{\phi}_{\vec{v}}$. □

Well-definedness of $\boldsymbol{\phi}$. The individual maps $\widetilde{\phi}_{\vec{v}}$ together define the map $\widetilde{\phi}$ as in the start of the section. In order to check that $\widetilde{\phi}$ descends to a well-defined homomorphism $\phi : \mathrm{Sp}_{2g}(\mathbb{Z})[2] \to \mathcal{Q}_g$, we need to check that $\widetilde{\phi}$ respects the edge and conjugation relations for $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ coming from its action on X_g. First we deal with the edge relations. As in Section 3.1.1, if $Y \in \mathrm{Sp}_{2g}(\mathbb{Z})[2]$ stabilizes the vertex $\vec{v}$ of X_g, then $Y_{\vec{v}}$ denotes the corresponding element of

$$\left(\mathrm{Sp}_{2g}(\mathbb{Z})[2]\right)_{\vec{v}} < \mathop{\ast}_{\vec{v} \in X_g^{(0)}} \left(\mathrm{Sp}_{2g}(\mathbb{Z})[2]\right)_{\vec{v}}.$$

Lemma 4.4. *If $\vec{v}, \vec{w} \in X_g^{(0)}$ are joined by an edge e and $Y \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_e$, then*

$$\widetilde{\phi}(Y_{\vec{v}}) = \widetilde{\phi}(Y_{\vec{w}}).$$

Proof. We will treat the case where $X_g = \widehat{\mathfrak{IB}}_g$. Recall that all simplices of additive type have dimension at least 2, and so we only need to consider standard edges and edges of intersection type. By removing all mention of edges of intersection type, we are left with a proof for the case where $X_g = \mathfrak{IB}_g$.

Before we begin in earnest, we need one bit of notation relating to the edges of X_g. Let c_{12} and c_{45} be the curves shown in Figure 2 and let $\vec{v}_{12}$ and $\vec{v}_{45}$ be defined in the same way as $\vec{v}_{23}$. Let $e_1 = \{\vec{v}_{23}, \vec{v}_{45}\}$ and $e_2 = \{\vec{v}_{23}, \vec{v}_{12}\}$. The edges e_1 is a standard edge in X_g and e_2 is an edge of intersection type.

The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on standard edges and on edges of intersection type in X_g. Interchanging $\vec{v}$ and $\vec{w}$ if necessary, there therefore exists some $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Z(\vec{v}_{23}) = \vec{v}$ and such that $Z(e_i) = e$ for some $1 \le i \le 2$. Also, there exists some $b \in \mathrm{B}_{2g+1}$ such that b interchanges the curves from Figure 2 corresponding to the endpoints of e_i; let $\beta = \widehat{\rho}(b) \in \widehat{\mathcal{Q}}_g$. We have that $Z\widehat{\pi}(\beta)(\vec{v}_{23}) = \pm\vec{w}$. Finally, since Y stabilizes e and $Z(e_i) = e$, it follows that $W = Z^{-1}YZ \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{e_i}$.

We will require Proposition 3.4(2) and one more naturality property, namely

$$\beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(W)\beta = \widetilde{\phi}_{\vec{v}_{23}}\left(\widehat{\pi}(\beta)^{-1}W\widehat{\pi}(\beta)\right). \tag{$*$}$$

The proof of $(*)$ is below. Assuming it for the moment, we are now ready to complete the proof that $\widetilde{\phi}(Y_{\vec{v}})$ is equal to $\widetilde{\phi}(Y_{\vec{w}})$:

$$\begin{aligned}\widetilde{\phi}(Y_{\vec{v}}) = \widetilde{\phi}_{\vec{v}}(Y) &= Z \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ) = Z\widehat{\pi}(\beta)\widehat{\pi}(\beta^{-1}) \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ) \\ &= Z\widehat{\pi}(\beta) \cdot \left(\beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ)\beta\right) = Z\widehat{\pi}(\beta) \cdot \widetilde{\phi}_{\vec{v}_{23}}(\widehat{\pi}(\beta)^{-1}Z^{-1}YZ\widehat{\pi}(\beta)) \\ &= \widetilde{\phi}_{\pm\vec{w}}(Y) = \widetilde{\phi}(Y_{\vec{w}}).\end{aligned}$$

In the fourth equality we used Proposition 3.4(2), in the fifth equality we used Equation $(*)$, and in the last equality we used Lemma 4.3. The other equalities simply use the definitions of $\widetilde{\phi}$ and $\widetilde{\phi}_{\vec{v}}$.

We now verify Equation $(*)$. In the course of constructing $\widetilde{\phi}$ we showed that $\pi|_{\Omega_{23}} : \Omega_{23} \to (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}_{23}}$ is surjective, and hence an isomorphism. In a similar vein, π restricts to isomorphisms

$$\begin{aligned}&\rho\left((\mathrm{PB}_{2g+1})_{\{c_{23},c_{45}\}}\right) \to (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{e_1}, \text{ and} \\ &\rho\left((\mathrm{PB}_{2g+1})_{\{c_{23},c_{12}\}}\right) \to (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{e_2}.\end{aligned}$$

The nontrivial part here is the surjectivity, which was again proven by the first two authors [11]. In particular, $\widetilde{\phi}_{\vec{v}_{23}}(W)$ lies in either $\rho\left((\mathrm{PB}_{2g+1})_{\{c_{23},c_{45}\}}\right)$ or $\rho\left((\mathrm{PB}_{2g+1})_{\{c_{23},c_{12}\}}\right)$; in other words, $\widetilde{\phi}_{\vec{v}_{23}}(W) = \rho(a)$, where a lies in either $(\mathrm{PB}_{2g+1})_{\{c_{23},c_{45}\}}$ or $(\mathrm{PB}_{2g+1})_{\{c_{23},c_{12}\}}$. Because b swaps c_{23} with either c_{45} or c_{12}, the braid $b^{-1}ab$ lies in $(\mathrm{PB}_{2g+1})_{\{c_{23}\}}$, and so $\beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(W)\beta$ lies in Ω_{23}. Using the fact that $\pi \circ \widetilde{\phi}_{\vec{v}_{23}} = \mathrm{id}$, we have

$$\pi\left(\beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(W)\beta\right) = \widehat{\pi}\left(\beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(W)\beta\right) = \widehat{\pi}\left(\beta^{-1}\right)\pi\left(\widetilde{\phi}_{\vec{v}_{23}}(W)\right)\widehat{\pi}(\beta) = \widehat{\pi}(\beta)^{-1}W\widehat{\pi}(\beta).$$

As $\widetilde{\phi}_{\vec{v}_{23}}$ and $\pi|_{\Omega_{23}}$ are inverses, we conclude

$$\widetilde{\phi}_{\vec{v}_{23}}\left(\widehat{\pi}(\beta)^{-1}W\widehat{\pi}(\beta)\right) = \beta^{-1}\widetilde{\phi}_{\vec{v}_{23}}(W)\beta.$$

This completes the proof. $\square$

Next we check that $\widetilde{\phi}$ respects the conjugation relations of $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$.

Lemma 4.5. *For any $Y \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{v}}$ and $U \in (\mathrm{Sp}_{2g}(\mathbb{Z})[2])_{\vec{w}}$, we have*

$$\widetilde{\phi}(U_{\vec{w}}Y_{\vec{v}}U_{\vec{w}}^{-1}) = \widetilde{\phi}\left((UYU^{-1})_{U(\vec{v})}\right).$$

Proof. Choose $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Z(\vec{v}_{23}) = \vec{v}$. We have

$$\begin{aligned}\widetilde{\phi}(U_{\vec{w}}Y_{\vec{v}}U_{\vec{w}}^{-1}) &= \widetilde{\phi}_{\vec{w}}(U)\widetilde{\phi}_{\vec{v}}(Y)\widetilde{\phi}_{\vec{w}}(U)^{-1} = \widehat{\pi}(\widetilde{\phi}_{\vec{w}}(U)) \cdot \widetilde{\phi}_{\vec{v}}(Y) = \pi(\widetilde{\phi}_{\vec{w}}(U)) \cdot \widetilde{\phi}_{\vec{v}}(Y)\\ &= U\cdot\widetilde{\phi}_{\vec{v}}(Y) = UZ \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}YZ) = UZ \cdot \widetilde{\phi}_{\vec{v}_{23}}(Z^{-1}U^{-1}(UYU^{-1})UZ)\\ &= \widetilde{\phi}_{U(\vec{v})}(UYU^{-1}) = \widetilde{\phi}\left((UYU^{-1})_{U(\vec{v})}\right).\end{aligned}$$

The second equality uses Proposition 3.4(2) and the fourth equality uses Lemma 4.2. Again, the remaining equalities use the definitions of $\widetilde{\phi}$ and $\widetilde{\phi}_{\vec{v}}$. $\square$

Completing the proof. It follows from Lemmas 4.1, 4.4, and 4.5 that $\phi : \mathrm{Sp}_{2g}(\mathbb{Z})[2] \to \mathcal{Q}_g$ is a well-defined homomorphism. It remains to check that ϕ is a left inverse of the projection map $\pi : \mathcal{Q}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})[2]$.

Lemma 4.6. $\phi \circ \pi = 1$.

Proof. The group PB_{2g+1} is generated by the conjugates in B_{2g+1} of the Dehn twist $T_{c_{23}}$; see Section 6.2.1 below. Therefore, $\mathcal{Q}_g$ is generated by elements of the form

$$\eta_b = \widehat{\rho}(b)\rho(T_{c_{23}})\widehat{\rho}(b)^{-1}$$

where $b \in \mathrm{B}_{2g+1}$. Thus, it suffices to check that $\phi(\pi(\eta_b)) = \eta_b$, where b is an arbitrary element of B_{2g+1}. Note that since $\pi(\rho(T_{c_{23}}))$ stabilizes $\vec{v}_{23}$, the conjugate $\pi(\eta_b)$ stabilizes $\vec{v}_b = Z_b(\vec{v}_{23})$ where $Z_b = \widehat{\pi}(\widehat{\rho}(b))$. We have

$$\begin{aligned}\phi(\pi(\eta_b)) &= \widetilde{\phi}((\pi(\eta_b))_{\vec{v_b}}) = \widetilde{\phi}_{\vec{v_b}}(\pi(\eta_b)) = \widetilde{\phi}_{\vec{v_b}}(\pi(\widehat{\rho}(b)\rho(T_{c_{23}})\widehat{\rho}(b)^{-1})) = \widetilde{\phi}_{\vec{v_b}}(\pi(Z_b \cdot \rho(T_{c_{23}})))\\ &= \widetilde{\phi}_{\vec{v_b}}(Z_b\pi(\rho(T_{c_{23}}))Z_b^{-1}) = Z_b \cdot \widetilde{\phi}_{\vec{v_{23}}}(Z_b^{-1}Z_b\pi(\rho(T_{c_{23}}))Z_b^{-1}Z_b) = Z_b \cdot \widetilde{\phi}_{\vec{v_{23}}}(\pi(\rho(T_{c_{23}})))\\ &= Z_b \cdot (\rho(T_{c_{23}})) = \widehat{\rho}(b)\rho(T_{c_{23}})\widehat{\rho}(b)^{-1} = \eta_b.\end{aligned}$$

The fourth and ninth equalities use Proposition 3.4(2). The fifth equality uses Proposition 3.4(1). The eighth equality uses the fact that $\pi|_{\Omega_{23}}$ and $\widetilde{\phi}_{\vec{v}_{23}}$ are inverses and the fact that $\rho(T_{c_{23}}) \in \Omega_{23}$. All other equalities simply use the definition of ϕ, $\widetilde{\phi}$, and $\widetilde{\phi}_{\vec{v}}$. $\square$

This completes the proof of Proposition 2.1. It now remains to prove Proposition 3.3 and 3.4, which we take care of in the remaining two sections.

5 Connectivity of the complex of isotropic bases

In this section, we prove Proposition 3.3, which gives that the complex of isotopic bases $\mathfrak{IB}_g(R)$ is $(g-2)$-connected if R is $\mathbb{Z}$ or a field and that the augmented complexes $\widehat{\mathfrak{IB}}_3(\mathbb{Z})$ and $\widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ are 1-connected and 2-connected, respectively. Before we begin, we recall some basic generalities about posets.

5.1 Posets

Let P be a poset. Consider $p \in P$. The *height* of p, denoted $\mathrm{ht}(p)$, is the largest number k such that there exists a strictly increasing chain

$$p_0 < p_1 < \cdots < p_k = p.$$

We will denote by $P_{>p}$ be the subposet of P consisting of elements strictly greater than p. Also, if $f : Q \to P$ is a poset map, then

$$f/p = \{q \in Q \mid f(q) \leq p\}.$$

Finally, the *geometric realization* of P, denoted $|P|$, is the simplicial complex whose vertices are elements of P and whose k-simplices are sets $\{p_0, \ldots, p_k\}$ of elements of P satisfying

$$p_0 < p_1 < \cdots < p_k.$$

A key example is as follows. Let X be a simplicial complex. Then the set $\mathfrak{P}(X)$ of simplices of X forms a poset under inclusion and $|\mathfrak{P}(X)|$ is the barycentric subdivision of X. When we say that a poset has some topological property, we really mean that its geometric realization has that property.

5.2 Connectivity of $\mathfrak{IB}_g(R)$

We first prove Proposition 3.3(1), which asserts that $\mathfrak{IB}_g(R)$ is homotopy equivalent to a wedge of $(g-1)$-spheres for R equal to $\mathbb{Z}$ or a field. The starting point is the following version of Quillen's Theorem A [38, Theorem 9.1].

Theorem 5.1. *Let $f : Q \to P$ be a poset map. For some m, assume that P is homotopy equivalent to a wedge of m-spheres. Also, for all $p \in P$ assume that $P_{>p}$ is homotopy equivalent to a wedge of $(m - ht(p) - 1)$-spheres and that f/p is homotopy equivalent to a wedge of $ht(p)$-spheres. Then Q is homotopy equivalent to a wedge of m-spheres.*

In our application of Theorem 5.1, we will take Q to be $\mathfrak{IB}_g(R)$ and P to be $\mathfrak{T}_g(R)$, which will be defined momentarily. Theorems 5.2 and 5.3 below give that $\mathfrak{IB}_g(R)$ and $\mathfrak{T}_g(R)$ (and the natural map between them) satisfy the hypotheses of Theorem 5.1 with $m = g-1$, and so we will conclude that $\mathfrak{IB}_g(R)$ is a wedge of $(g-1)$-spheres, as desired.

Buildings. Let $\mathbb{F}$ be a field and $\hat{\imath}$ the standard symplectic form on $\mathbb{F}^{2g}$. An isotropic subspace of $\mathbb{F}^{2g}$ is a subspace on which $\hat{\imath}$ vanishes. The Tits building $\mathfrak{T}_g(\mathbb{F})$ is the poset of nontrivial isotropic subspaces of $\mathbb{F}^{2g}$. The key theorem about the topology of $\mathfrak{T}_g(\mathbb{F})$ is the Solomon–Tits theorem [13, Theorem 4.73].

Theorem 5.2 (Solomon–Tits)**.** *If $\mathbb{F}$ is a field, then $\mathfrak{T}_g(\mathbb{F})$ is homotopy equivalent to a wedge of $(g-1)$-spheres. Also, for $V \in \mathfrak{T}_g(\mathbb{F})$ the poset $(\mathfrak{T}_g(\mathbb{F}))_{>V}$ is homotopy equivalent to a wedge of $(g-2-ht(V))$-spheres.*

Complexes of partial bases. For R equal to either $\mathbb{Z}$ or a field, let $\mathfrak{B}_n(R)$ be the simplicial complex whose k-simplices are sets $\{\vec{v}_0, \ldots, \vec{v}_k\}$ of vectors in R^n that form a basis for a free summand of R^n. The following theorem appears in Maazen's thesis [25].

Theorem 5.3. *If R is either $\mathbb{Z}$ or a field, then $\mathfrak{B}_n(R)$ is homotopy equivalent to a wedge of $(n-1)$-spheres.*

For $R = \mathbb{Z}$, there is a published account of Theorem 5.3 in [16, Step 2, Proof of Theorem B]. This proof can be easily adapted to prove Theorem 5.3 for R a field. Instead of using the Euclidean algorithm to decrease the absolute value of the last coordinate of a vector, one can simply make the last coordinate 0 by using the inverses provided by the field.

Proof of Proposition 3.3(1). Let $\mathbb{F} = R$ if R is a field and $\mathbb{F} = \mathbb{Q}$ if $R = \mathbb{Z}$. Define a poset map

$$span : \mathfrak{P}(\mathfrak{I}\mathfrak{B}_g(R)) \to \mathfrak{T}_g(\mathbb{F})$$

by $span(\{\vec{v}_0, \ldots, \vec{v}_k\}) = \operatorname{span}(\vec{v}_0, \ldots, \vec{v}_k)$. Consider $V \in \mathfrak{T}_g(\mathbb{F})$, and set $d = \dim(V)$, so $\operatorname{ht}(V) = \dim(V) - 1$. The poset $span/V$ is isomorphic to $\mathfrak{B}_d(V)$, so Theorem 5.3 says that $span/V$ is homotopy equivalent to a wedge of $\operatorname{ht}(V)$-spheres. Theorem 5.2 says that the remaining assumptions of Theorem 5.1 are satisfied for $span$ with $m = g-1$. The conclusion of this theorem is that $\mathfrak{P}(\mathfrak{I}\mathfrak{B}_g(R))$, hence $\mathfrak{I}\mathfrak{B}_g(R)$, is a wedge of $(g-1)$-spheres, as desired. $\square$

5.3 Connectivity of $\widehat{\mathfrak{I}\mathfrak{B}}_g(R)$

We now prove Proposition 3.3(2), which asserts that the complexes $\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z})$ and $\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z}/2)$ are 1-connected and 2-connected, respectively. The proof is more complicated than the one for Proposition 3.3(1), and so we begin with an outline.

For R either $\mathbb{Z}$ or a field, denote by $\mathfrak{I}\mathfrak{B}_g^{\alpha}(R)$ the subcomplex of $\widehat{\mathfrak{I}\mathfrak{B}}_g(R)$ consisting of standard simplices and simplices of additive type. First let us explain the most difficult part, that $\pi_2(\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z}/2)) = 1$. The steps are:

1. We show that the map $\pi_k(\mathfrak{I}\mathfrak{B}_3^{\alpha}(R)) \to \pi_k(\widehat{\mathfrak{I}\mathfrak{B}}_3(R))$ is surjective for $k = 1, 2$.
2. We find an explicit generating set for $\pi_2(\mathfrak{I}\mathfrak{B}_3^{\alpha}(\mathbb{Z}/2))$.
3. We show that each generator of $\pi_2(\mathfrak{I}\mathfrak{B}_3^{\alpha}(\mathbb{Z}/2))$ maps to zero in $\pi_2(\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z}/2))$.

The other two statements, namely, the 1-connectivity of $\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z})$ and $\widehat{\mathfrak{I}\mathfrak{B}}_3(\mathbb{Z}/2)$, follow from the first step, as $\mathfrak{I}\mathfrak{B}_3(R)$ is 1-connected (Proposition 3.3(1)) and $\mathfrak{I}\mathfrak{B}_3(R)$ contains the entire 1-skeleton of $\mathfrak{I}\mathfrak{B}_3^{\alpha}(R)$.

Pushing into $\mathfrak{I}\mathfrak{B}_3^{\alpha}(R)$. As above, the first step of the proof is to show that $\mathfrak{I}\mathfrak{B}_g^{\alpha}(R)$ carries all of $\pi_1(\widehat{\mathfrak{I}\mathfrak{B}}_g(R))$ and $\pi_2(\widehat{\mathfrak{I}\mathfrak{B}}_g(R))$. We in fact have a more general statement.

Lemma 5.4. *Let R equal $\mathbb{Z}$ or a field. For $1 \leq k \leq g-1$, the inclusion map $\mathfrak{IB}_g^\alpha(R) \to \widehat{\mathfrak{IB}}_g(R)$ induces a surjection on π_k.*

Proof. Let S be a simplicial complex that is a combinatorial triangulation of a k-sphere and let $f : S \to \widehat{\mathfrak{IB}}_g(R)$ be a simplicial map. It is enough to homotope f such that its image lies in $\mathfrak{IB}_g^\alpha(R)$. If the image of f is not contained in $\mathfrak{IB}_g^\alpha(R)$, then there exists some simplex σ of S such that $f(\sigma)$ is a 1-simplex $\{\vec{a}, \vec{b}\}$ of intersection type. Choose σ such that $d = \dim(\sigma)$ is maximal; since f need not be injective, we might have $d > 1$. The link $\mathrm{Link}_S(\sigma)$ is homeomorphic to a $(k-d-1)$-sphere (recall that the link of σ is union of simplices in the closed star of σ that do not contain σ). Moreover,

$$f(\mathrm{Link}_S(\sigma)) \subset \mathrm{Link}_{\widehat{\mathfrak{IB}}_g(R)}\{\vec{a}, \vec{b}\}.$$

The key observation is that $\mathrm{Link}_{\widehat{\mathfrak{IB}}_g(R)}\{\vec{a}, \vec{b}\}$ can only contain standard simplices, and moreover all of its vertices are vectors $\vec{v}$ such that $\hat{\imath}(\vec{a}, \vec{v}) = \hat{\imath}(\vec{b}, \vec{v}) = 0$. The $\hat{\imath}$-orthogonal complement of $\mathrm{span}(\vec{a}, \vec{b})$ is isomorphic to a $(g-1)$-dimensional free symplectic module over R, so we deduce that

$$\mathrm{Link}_{\widehat{\mathfrak{IB}}_g(R)}\{\vec{a}, \vec{b}\} \cong \mathfrak{IB}_{g-1}(R).$$

Proposition 3.3(1) says that $\mathfrak{IB}_{g-1}(R)$ is $(g-3)$-connected. Since $k - d - 1 \leq g - 3$, we conclude that there exists a combinatorially triangulated $(k-d)$-ball B with $\partial B = \mathrm{Link}_S(\sigma)$ and a simplicial map

$$F : B \to \mathrm{Link}_{\widehat{\mathfrak{IB}}_g(R)}\{\vec{a}, \vec{b}\}$$

such that $F|_{\partial B} = f|_{\mathrm{Link}_S(\sigma)}$. We can therefore homotope f so as to replace $f|_{\mathrm{Star}_S(\sigma)}$ with $F|_B$. This eliminates σ without introducing any new d-dimensional simplices mapping to 1-simplices of intersection type. Doing this repeatedly homotopes f such that its image contains no simplices of intersection type. □

Generators for $\boldsymbol{\pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2))}$. Our next goal is to give generators for $\pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2))$. This has two parts. We first give an explicit generating set for $\pi_2(\mathfrak{T}_g(\mathbb{Z}/2))$ (Theorem 5.5), and then we show that the map

$$\mathit{span} : \mathfrak{P}(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2)) \to \mathfrak{T}_3(\mathbb{Z}/2)$$

given by $\mathit{span}(\{\vec{v}_0, \ldots, \vec{v}_k\}) = \mathrm{span}(\vec{v}_0, \ldots, \vec{v}_k)$ induces an isomorphism on π_2 (Lemma 5.8); thus the generators for $\pi_2(\mathfrak{T}_g(\mathbb{Z}/2))$ give generators for $\pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2))$.

Let $\mathbb{F}$ be a field. Recall that the Solomon–Tits theorem (Theorem 5.2) says that the Tits building $\mathfrak{T}_g(\mathbb{F})$ is homotopy equivalent to a wedge of $(g-1)$-spheres. The next theorem gives explicit generators for $\pi_{g-1}(\mathfrak{T}_g(\mathbb{F}))$. First, we require some setup.

Let Y_g be the join of g copies of S^0, so $Y_g \cong S^{g-1}$. If x_i and y_i are the vertices of the i^{th} copy of S^0 in Y_g, then the simplices of Y_g are the subsets $\sigma \subset \{x_1, y_1, \ldots, x_g, y_g\}$ such that σ contains at most one of x_i and y_i for each $1 \leq i \leq 2g$. Given a symplectic basis $B = \{\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_g, \vec{b}_g\}$ for $\mathbb{F}^{2g}$, we obtain a poset map $\alpha_B : \mathfrak{P}(Y_g) \to \mathfrak{T}_g(\mathbb{F})$ as follows. Consider

$$\sigma = \{x_{i_1}, \ldots, x_{i_k}, y_{j_1}, \ldots, y_{j_\ell}\} \in \mathfrak{P}(Y_g).$$

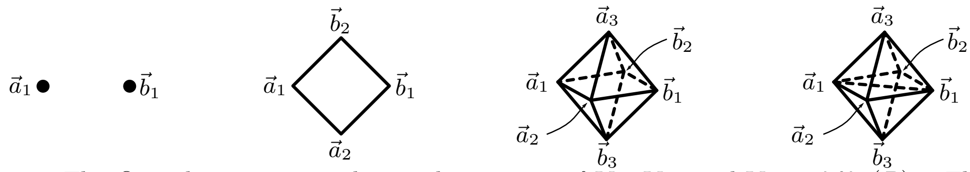

Figure 3: *The first three pictures depict the images of Y_1, Y_2, and Y_3 in $\mathfrak{IB}_g(R)$. The fourth picture indicates a nullhomotopy of Y_3 in $\widehat{\mathfrak{IB}}_g(R)$ using simplices of intersection type.*

We then define

$$\alpha_B(\sigma) = \mathrm{span}(\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}, \vec{b}_{j_1}, \ldots, \vec{b}_{j_\ell}) \in \mathfrak{T}_g(\mathbb{F}).$$

The resulting map $\alpha_B : Y_g \to \mathfrak{T}_g(\mathbb{F})$ is a $(g-1)$-sphere in $\mathfrak{T}_g(\mathbb{F})$. We have the following theorem [13, Theorem 4.73].

Theorem 5.5. *The group $\pi_{g-1}(\mathfrak{T}_g(\mathbb{F}))$ is generated by the set*

$$\{[\alpha_B] \in \pi_{g-1}(\mathfrak{T}_g(\mathbb{F})) \mid B \text{ a symplectic basis for } \mathbb{F}^{2g}\}.$$

Now, in the same way as we defined the $\alpha_B(\sigma)$, we can also define

$$\widetilde{\alpha}_B : Y_3 \to \mathfrak{IB}_3(\mathbb{Z}/2)$$

via

$$\widetilde{\alpha}_B(\sigma) = \{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}, \vec{b}_{j_1}, \ldots, \vec{b}_{j_\ell}\} \in \mathfrak{P}(\mathfrak{IB}_3(\mathbb{Z}/2));$$

see Figure 3. We have

$$\alpha_B = span \circ \widetilde{\alpha}_B.$$

We will show $span_* : \pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2)) \to \pi_2(\mathfrak{T}_3(\mathbb{Z}/2))$ is an isomorphism (Lemma 5.8), and hence the $\widetilde{\alpha}_B(\sigma)$ generate $\pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2))$ (Lemma 5.9). The starting point here is another version of Quillen's Theorem A [8, Theorem 2].

Theorem 5.6. *Let Q and P be connected posets and $f : Q \to P$ a poset map. Assume that f/p is m-connected for all $p \in P$. Then the induced map $f_* : \pi_k(Q) \to \pi_k(P)$ is an isomorphism for $1 \leq k \leq m$.*

We will also need the following easy lemma.

Lemma 5.7. *Any subset of $(\mathbb{Z}/2)^n$ with cardinality at most 4 has one of the forms:*

$$\{\}, \{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_1 + v_2\},$$
$$\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_1 + v_2\}, \{v_1, v_2, v_3, v_1 + v_2 + v_3\},$$

where in each set the v_i are linearly independent vectors in $(\mathbb{Z}/2)^n$.

Lemma 5.8. *The map $span_* : \pi_2(\mathfrak{IB}_3^\alpha(\mathbb{Z}/2)) \to \pi_2(\mathfrak{T}_3(\mathbb{Z}/2))$ is an isomorphism.*

Proof. Consider $V \in \mathfrak{T}_3(\mathbb{Z}/2)$, and let $d = \dim(V) \leq 3$, so $\mathrm{ht}(V) = d - 1$. The poset $span/V$ is isomorphic to the result $\mathfrak{B}_d^\alpha(\mathbb{Z}/2)$ of attaching the analogues of cells of additive type to $\mathfrak{B}_d(\mathbb{Z}/2)$. Lemma 5.7 implies that the 3-skeleton of $\mathfrak{B}_d^\alpha(\mathbb{Z}/2)$ contains all subsets of vertices of $\mathfrak{B}_d^\alpha(\mathbb{Z}/2)$ of size at most 4. In particular, $\mathfrak{B}_d^\alpha(\mathbb{Z}/2)$ is 2-connected. The lemma now follows from Theorem 5.6. ☐

We have the following immediate consequence of Theorem 5.5 and Lemma 5.8.

Lemma 5.9. *As B ranges over all symplectic bases, the homotopy classes of the maps*

$$Y_3 \xrightarrow{\widetilde{\alpha}_B} \mathfrak{IB}_3(\mathbb{Z}/2) \hookrightarrow \mathfrak{IB}_3^{\alpha}(\mathbb{Z}/2)$$

generate $\pi_2(\mathfrak{IB}_3^{\alpha}(\mathbb{Z}/2))$.

Proof of Proposition 3.3(2). We already explained how the 1-connectivity of $\widehat{\mathfrak{IB}}_3(\mathbb{Z})$ and of $\widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ follow from Lemma 5.4. It remains to show that $\widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ is 2-connected. Lemma 5.4 says that the inclusion map $\mathfrak{IB}_3^{\alpha}(\mathbb{Z}/2) \hookrightarrow \widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ induces a surjection on π_2. We will prove that it induces the zero map as well.

By Lemma 5.9, it suffices to show that, for each symplectic basis B of $(\mathbb{Z}/2)^6$, the map

$$Y_3 \xrightarrow{\widetilde{\alpha}_B} \mathfrak{IB}_3(\mathbb{Z}/2) \hookrightarrow \mathfrak{IB}_3^{\alpha}(\mathbb{Z}/2) \hookrightarrow \widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$$

is nullhomotopic. In the fourth picture in Figure 3, we indicate an explicit nullhomotopy of $\widetilde{\alpha}_B(Y_3)$ in $\widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$. Specifically, we realize $\widetilde{\alpha}_B(Y_3)$ as the boundary of a 3-ball formed by four simplices of intersection type. We conclude that the inclusion $\mathfrak{IB}_3^{\alpha}(\mathbb{Z}/2) \hookrightarrow \widehat{\mathfrak{IB}}_3(\mathbb{Z}/2)$ induces the zero map on π_2, as desired. $\square$

6 The symplectic group action

In this section, we prove Proposition 3.4. Recall that as part of the hypotheses, we are assuming $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for $h < g$. This inductive assumption will be in place throughout this section. Also, in this section we will use the notation $\mathcal{Q}_g = \mathrm{PB}_{2g+1}/\Theta_{2g+1}$ and $\widehat{\mathcal{Q}}_g = \mathrm{B}_{2g+1}/\Theta_{2g+1}$. Finally, we will again denote by $\rho : \mathrm{PB}_{2g+1} \to \mathcal{Q}_g$, $\widehat{\rho} : \mathrm{B}_{2g+1} \to \mathcal{Q}_g$, $\pi : \mathcal{Q}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})[2]$, and $\widehat{\pi} : \widehat{\mathcal{Q}}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ the natural maps.

6.1 Outline

We wish to construct an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ as in Proposition 3.4. We will start in Section 6.2 by giving finite presentations $\mathrm{Sp}_{2g}(\mathbb{Z}) = \langle S_{\mathrm{Sp}} \mid R_{\mathrm{Sp}} \rangle$ and $\mathcal{Q}_g = \langle S_{\mathcal{Q}} \mid R_{\mathcal{Q}} \rangle$. Our basic strategy is to propose an action by declaring where each element of S_{Sp} sends each element of $S_{\mathcal{Q}}$, and then check the proposed action respects all relations in R_{Sp} and $R_{\mathcal{Q}}$.

The trick here is to find just the right balance: more generators for $\mathrm{Sp}_{2g}(\mathbb{Z})$ will mean that checking the well-definedness of our action with respect to the $\mathrm{Sp}_{2g}(\mathbb{Z})$ relations is easier (relations are smaller), but checking the well-definedness with respect to the $\mathcal{Q}_g$ relations is harder (more cases to check), and vice versa.

Denote by $F(S_{\mathrm{Sp}})$ and $F(S_{\mathcal{Q}})$ the free groups on S_{Sp} and $S_{\mathcal{Q}}$. Also, for $w \in F(S_{\mathrm{Sp}})$ (resp. $w \in F(S_{\mathcal{Q}})$), write $\overline{w}$ for the corresponding element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (resp. $\mathcal{Q}_g$). For each $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $s \in S_{\mathcal{Q}}$, we construct in Section 6.3 an element $t \cdot s \in \mathcal{Q}_g$ that satisfies the following naturality property:

$$\pi(\overline{t \cdot s}) = \overline{t}\pi(\overline{s})\overline{t}^{-1}.$$

Our action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ will eventually be defined by $\overline{t} \cdot \overline{s} = t \cdot s$.

Homomorphisms $F(S_{\mathcal{Q}}) \to \mathcal{Q}_g$ are in bijection with functions $S_{\mathcal{Q}} \to \mathcal{Q}_g$. Therefore, for each $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$, there is a homomorphism $\hat{f}_t : F(S_{\mathcal{Q}}) \to \mathcal{Q}_g$ given by $\hat{f}_t(s) = t \cdot s$ for all $s \in S_{\mathcal{Q}}$. In Section 6.4 we prove the following lemma, which implies that each $\hat{f}_t$ descends to a homomorphism $f_t : \mathcal{Q}_g \to \mathcal{Q}_g$.

Lemma 6.1. *For all $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $r \in R_{\mathcal{Q}}$, we have $\hat{f}_t(r) = 1$.*

Denote by $\widehat{F}(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1})$ the free monoid on $S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$. There is a monoid homomorphism $\Phi : \widehat{F}(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}) \to \mathrm{End}(\mathcal{Q}_g)$ given by $\Phi(t) = f_t$ for all $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$. In other words, the monoid $\widehat{F}(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1})$ acts on $\mathcal{Q}_g$. The next lemma, proven in Section 6.5, shows that this monoid action descends to an action of the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$.

Lemma 6.2. *Regard R_{Sp} as a subset of $\widehat{F}(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1})$. Consider $r \in \widehat{F}(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1})$ such that either $r \in R_{\mathrm{Sp}}$ or $r = tt^{-1}$ for some $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$. We then have $\Phi(r) = 1$.*

In Section 6.6, we will verify that the resulting action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ also satisfies all three properties laid out in Proposition 3.4.

6.2 Presentations for $\mathcal{Q}_g$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$

Our entire strategy for constructing our action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$ rests on having well-chosen presentations for both groups. We will give generators for $\mathcal{Q}_g$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$ in Sections 6.2.1 and 6.2.2 and relations in Sections 6.2.3 and 6.2.4.

6.2.1 Generators for $\mathcal{Q}_g$

Since $\mathcal{Q}_g$ is a quotient of PB_{2g+1}, any set of generators for PB_{2g+1} descends to a set of generators for $\mathcal{Q}_g$. We identify PB_{2g+1} with the pure mapping class group of a disk D_{2g+1} with $2g+1$ marked points $p_1, \cdots, p_{2g+1}$, that is the group of homotopy classes of homeomorphisms of D_{2g+1} that fix each p_i and each point of the boundary; see [17, Section 9.3]. For concreteness, we take D_{2g+1} to be a convex Euclidean disk and the p_i to lie on the vertices of a regular $(2g+1)$-gon, appearing clockwise in cyclic order; see the left-hand side of Figure 4. Choose this identification so that the curve c_{23} in Figure 2 is the boundary of a convex region containing p_2 and p_3 and no other p_i.

More generally, for any subset A of $\{1, \ldots, 2g+1\}$ we denote by c_A the simple closed curve in D_{2g+1} that bounds a convex region of D_{2g+1} containing precisely $\{p_i \mid i \in A\}$ in its interior; this curve is unique up to homotopy in D_{2g+1}. We will write c_{ij} or $c_{i,j}$ for $c_{\{i,j\}}$, etc., when convenient. The curves c_{1234} and c_{45} are shown in Figure 4.

Artin proved that PB_{2g+1} is generated by the Dehn twists about the curves in the set

$$\mathcal{C}(S_Q) = \{c_{ij} \mid 1 \leq i < j \leq 2g+1\}.$$

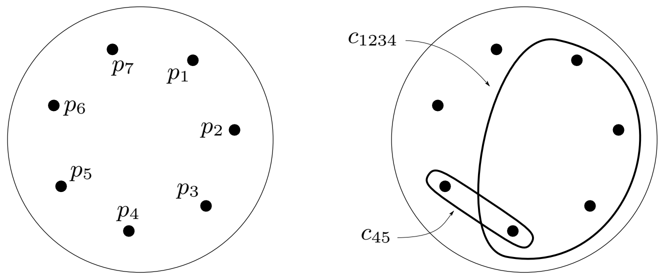

Figure 4: *Left: The disk with* D_{2g+1}*, with its marked points arranged clockwise on the vertices of a convex* $(2g+1)$*-gon. Right: Two convex simple closed curves.*

The resulting generating set for $\mathcal{Q}_g$ is

$$S_{\mathcal{Q}} = \{s_c : c \in \mathcal{C}(S_{\mathcal{Q}})\},$$

where for any simple closed curve c in D_{2g+1}, the symbol s_c represents $\rho(T_c)$, the image in $\mathcal{Q}_g$ of the Dehn twist about c.

6.2.2 Generators for $\mathrm{Sp}_{2g}(\mathbb{Z})$

The *transvection* on $v \in \mathbb{Z}^{2g}$ is the element $\tau_v \in \mathrm{Sp}_{2g}(\mathbb{Z})$ given by

$$\tau_v(w) = w + \hat{\imath}(w, v)\, v \qquad (w \in \mathbb{Z}^{2g}),$$

where $\hat{\imath}$ is the symplectic form. Note that $\tau_v = \tau_{-v}$. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by transvections on primitive elements of $\mathbb{Z}^{2g}$.

The transvections on primitive elements of $\mathbb{Z}^{2g}$ are precisely the images of Dehn twists about nonseparating curves under the surjection $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z})$, so we can associate transvections to simple closed curves. Specifically, consider a simple closed curve a in D_{2g+1} surrounding an even number of marked points. The preimage of a in Σ_g^1 is a pair of disjoint nonseparating simple closed curves $\tilde{a}_1$ and $\tilde{a}_2$ that are homologous (after choosing consistent orientations), and so there is a corresponding transvection $\tau_{[\tilde{a}_1]} = \tau_{[\tilde{a}_2]}$ associated to a. This is the unique transvection τ_v such that $\tau_v^2 = \pi(s_a)$ in $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$.

We pause now to record the following.

Lemma 6.3. *For a simple closed curve* c *in* D_{2g+1} *surrounding an even number of marked points, we have*

$$\pi(\rho(T_c)) = \tau_{[\tilde{c}]}^2,$$

where $\tilde{c}$ *is one component of the preimage of* c *in* Σ_g^1*.*

Proof. We must determine the image of T_c under

$$\mathrm{B}_{2g+1} \xrightarrow{L} \mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z});$$

here L is the lifting map described in the introduction. The preimage in Σ_g^1 of c is the pair of simple closed curves $\tilde{c}$ and $\iota(\tilde{c})$, and so $L(T_c)$ is equal to $T_{\tilde{c}} T_{\iota(\tilde{c})}$. The curves $\tilde{c}$ and $\iota(\tilde{c})$ are homologous (up to sign) and so they have the same image in $\mathrm{Sp}_{2g}(\mathbb{Z})$, namely $\tau_{[\tilde{c}]}$. The lemma follows. $\square$

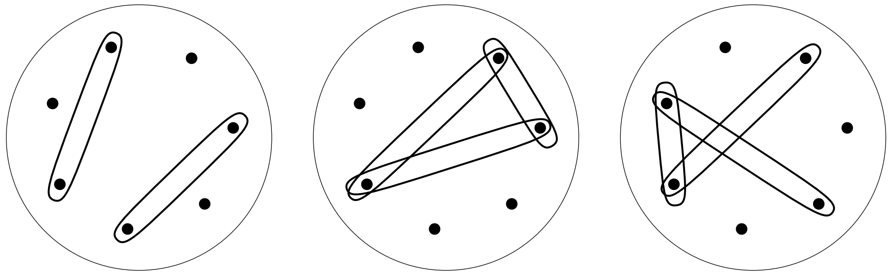

Figure 5: *The three configurations of curves used in the disjointness relations, the triangle relations, and the crossing relations for the pure braid group.*

We are now ready to list the specific curves a in D_{2g+1} giving generators t_a for $\mathrm{Sp}_{2g}(\mathbb{Z})$. We will continue to use the notation from Section 6.2.1. Denote by a_0 the convex simple closed curve c_{1234}. Also, for $1 \leq i \leq 2g$, set $a_i = c_{i,i+1}$. Humphries (see [17, Section 4]) proved that one can choose connected components $\tilde{a}_i$ of a_i in Σ_g^1 such that Mod_g^1 is generated by the Dehn twists about the curves $\tilde{a}_i$ (in fact, any set of choices will do). Since Mod_g^1 surjects onto $\mathrm{Sp}_{2g}(\mathbb{Z})$, it follows that the transvections associated to $a_0, a_1, \ldots, a_{2g}$ generate $\mathrm{Sp}_{2g}(\mathbb{Z})$.

In order to simplify our presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$ we need to add some auxiliary generators to $\mathrm{Sp}_{2g}(\mathbb{Z})$. Consider the following curves:

$$\begin{array}{lll} a_0' = c_{1245} & b_2 = c_{3456} & u' = c_{2345} \\ b_1 = c_{1256} & b_3 = c_{123456} & v = c_{134567} \\ b_1' = c_{2356} & u = c_{1267} & v' = c_{123467}. \end{array}$$

Let

$$\mathcal{C}(S_{Sp})_2 = \{a_1, \ldots, a_{2g}\},\ \mathcal{C}(S_{Sp})_4 = \{a_0, b_1, b_2, u, u'\},\ \mathcal{C}(S_{Sp})_6 = \{b_3, v, v'\},$$

and

$$\mathcal{C}(S_{Sp}) = \mathcal{C}(S_{Sp})_2 \cup \mathcal{C}(S_{Sp})_4 \cup \mathcal{C}(S_{Sp})_6.$$

The subscripts 2, 4, and 6 refer to the number of marked points that each curve surrounds. The resulting generating set for $\mathrm{Sp}_{2g}(\mathbb{Z})$ is

$$S_{Sp} = \{t_a \mid a \in \mathcal{C}(S_{Sp})\},$$

where t_a corresponds to the transvection associated to a as above.

6.2.3 Relations for $\mathcal{Q}_g$

Our set of relations $R_{\mathcal{Q}}$ for $\mathcal{Q}_g$ will consist of the four families of relations below. Recall that $\mathcal{Q}_g$ is defined as the quotient $\mathcal{Q}_g = \mathrm{PB}_{2g+1} / \Theta_{2g+1}$. We will first give a finite presentation for PB_{2g+1}, and then add relations for normal generators of Θ_{2g+1} inside PB_{2g+1}. There are many presentations for the pure braid group, most notably the original presentation due to Artin [3]. We will use here a modified version of Artin's presentation due to the second author and McCammond [27, Theorem 2.3]. The generating set is the generating set $S_{\mathcal{Q}}$ from Section 6.2.1. There are four types of defining relations for PB_{2g+1}, as follows.

1. Disjointness relations: $\quad [s_{c_{ij}}, s_{c_{rs}}] = 1$ if $i < j < r < s$ or
$i < r < s < j$

2. Triangle relations: $\quad s_{c_{ij}} s_{c_{jk}} s_{c_{ki}} = s_{c_{jk}} s_{c_{ki}} s_{c_{ij}} = s_{c_{ki}} s_{c_{ij}} s_{c_{jk}}$ if $i < j < k$

3. Crossing relations: $\quad [s_{c_{ij}}, s_{c_{js}} s_{c_{rs}} s_{c_{js}}^{-1}] = 1$ if $i < r < j < s$ or
$r < i < s < j$

We now add relations coming from Θ_{2g+1}. This group is normally generated in PB_{2g+1} by the squares of Dehn twists about the convex curves in D_{2g+1} surrounding odd numbers of marked points. We need to add one relation for each of these elements.

4. Odd twist relations: $\quad \Big((s_{c_{i_1 i_2}} \cdots s_{c_{i_1 i_n}}) \cdots (s_{c_{i_{n-2} i_{n-1}}} s_{c_{i_{n-2} i_n}}) s_{c_{i_{n-1} i_n}}\Big)^2 = 1$
for any $i_1 < \cdots < i_n$, where $3 \leq n \leq 2g+1$.

As elements of the pure braid group, we have:

$$(\overline{s}_{c_{i_1 i_2}} \cdots \overline{s}_{c_{i_1 i_n}}) \cdots (\overline{s}_{c_{i_{n-2} i_{n-1}}} \overline{s}_{c_{i_{n-2} i_n}}) \overline{s}_{c_{i_{n-1} i_n}} = \overline{s}_{c_{i_1} \cdots c_{i_n}};$$

see [17, Section 9.3].

6.2.4 Relations for $\mathrm{Sp}_{2g}(\mathbb{Z})$

Our set of relations R_{Sp} for $\mathrm{Sp}_{2g}(\mathbb{Z})$ will consist of the six families of relations below. Since $\mathrm{Sp}_{2g}(\mathbb{Z}) \cong \mathrm{Mod}_g^1 / \mathcal{I}_g^1$, we obtain a presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$ by starting with a presentation for Mod_g^1 and adding one relation for each normal generator of $\mathcal{I}_g^1$ in Mod_g^1. Wajnryb [39, 7] gave a finite presentation for Mod_g^1 with generating set $\{T_{\tilde{a}_0}, \ldots, T_{\tilde{a}_{2g}}\}$, where $T_{\tilde{a}_i}$ is the Dehn twist about one particular component of the preimage of a_i in Σ_g^1. The image of $T_{\tilde{a}_i}$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is t_{a_i} and so we obtain the first part of our presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$ by replacing each $T_{\tilde{a}_i}$ in Wajnryb's presentation with t_{a_i}. We have the following list of relations; see [17, Theorem 5.3]. Here $i(\cdot,\cdot)$ denotes the geometric intersection number of two curves.

1. Disjointness relations: $\quad t_{a_i} t_{a_j} = t_{a_j} t_{a_i}$ if $i(a_i, a_j) = 0$

2. Braid relations: $\quad t_{a_i} t_{a_{i+1}} t_{a_i} = t_{a_{i+1}} t_{a_i} t_{a_{i+1}}$ if $i(a_i, a_j) = 1$

3. 3-chain relation: $\quad (t_{a_1} t_{a_2} t_{a_3})^4 = t_{a_0} t_{b_0}$, where

$$t_{b_0} = (t_{a_4} t_{a_3} t_{a_2} t_{a_1} t_{a_1} t_{a_2} t_{a_3} t_{a_4}) t_{a_0} (t_{a_4} t_{a_3} t_{a_2} t_{a_1} t_{a_1} t_{a_2} t_{a_3} t_{a_4})^{-1}$$

4. Lantern relation: $\quad t_{a_0} t_{b_2} t_{b_1} = t_{a_1} t_{a_3} t_{a_5} t_{b_3}$

In the above relations, we have replaced some complicated expressions from Wajnryb's relations with some of our auxiliary generators. Thus, we need to add relations that express each of these generators in terms of the t_{a_i}.

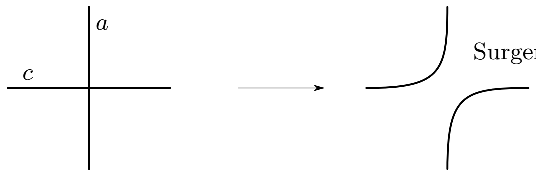

Figure 6: *Surgery on curves in* D_{2g+1}.

5. Auxiliary relations:

$$
\begin{array}{rl}
(i) & t_{a'_0} = (t_{a_4}t_{a_3})^{-1}t_{a_0}(t_{a_4}t_{a_3}) \\
(ii) & t_{b_1} = (t_{a_5}t_{a_4})^{-1}t_{a'_0}(t_{a_5}t_{a_4}) \\
(iii) & t_{b'_1} = (t_{a_2}t_{a_1})^{-1}t_{b_1}(t_{a_2}t_{a_1}) \\
(iv) & t_{b_2} = (t_{a_3}t_{a_2})^{-1}t_{b'_1}(t_{a_3}t_{a_2}) \\
(v) & t_u = (t_{a_6}t_{a_5})^{-1}t_{b_1}(t_{a_6}t_{a_5}) \\
(vi) & t_{u'} = (t_{a_4}t_{a_3}t_{a_2}t_{a_1})^{-1}t_{a_0}(t_{a_4}t_{a_3}t_{a_2}t_{a_1}) \\
(vii) & t_v = t_u t_{u'} t_u^{-1} \\
(viii) & t_{v'} = (t_{a_4}t_{a_3}t_{a_2})t_v(t_{a_4}t_{a_3}t_{a_2})^{-1} \\
(ix) & t_{b_3} = (t_{a_6}t_{a_5})t_{v'}(t_{a_6}t_{a_5})^{-1}
\end{array}
$$

By work of Johnson [21], the group $\mathcal{I}_g^1$ is normally generated by the single element $a_0b_0^{-1}$ when $g \geq 3$. Thus, to obtain our presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$, we simply add one more relation.

6. Bounding pair relation: $t_{a_0} = t_{b_0}$

6.3 Construction of the action

Let $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $x \in S_{\mathcal{Q}}$. The goal of this section is to construct an element $t \cdot x \in \mathcal{Q}_g$ that satisfies the naturality property

$$\pi\left(\overline{t \cdot x}\right) = \overline{t}\pi(\overline{x})\overline{t}^{-1}. \tag{1}$$

For a transvection $\tau_w \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and a square of a transvection $\tau_v^2 \in \mathrm{Sp}_{2g}(\mathbb{Z})[2]$, we have

$$\tau_w \tau_v^2 \tau_w^{-1} = \tau^2_{\tau_w(v)}. \tag{2}$$

Since transvections generate $\mathrm{Sp}_{2g}(\mathbb{Z})$ and squares of transvections generate $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$, the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is completely described by this formula. If we write $w(v)$ for $\tau_w(v)$, then this formula becomes $\tau_w \tau_v^2 \tau_w^{-1} = \tau^2_{w(v)}$. In other words, the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is given by an "action" of $\mathbb{Z}^{2g}$ on itself. Our strategy is to give an analogous action of the set of curves in D_{2g+1} on itself, and use this to define each $t_a \cdot s_c$.

First, for two simple closed curves a and c in D_{2g+1} we define

$$|\hat{\imath}|(a, c)$$

to be the absolute value of the algebraic intersection number of any two connected components of the preimages of a and c in Σ_g^1. These curves do not have a canonical orientation, so the algebraic intersection is not itself well defined.

Next let na denote n parallel copies of the curve a. We define Surger(na, c) to be the collection of simple closed curves obtained from $na \cup c$ by performing the surgery shown in Figure 6 at each point of $na \cap c$ and then discarding any inessential components. Note that this definition does not depend on any orientations of a or c. Also note that, by the bigon criterion (see [17, Section 1.2.4]), the homotopy class of Surger(na, c) only depends on n and the homotopy classes of a and c.

We now give our "action" of $\mathcal{C}(S_{Sp})$ on $\mathcal{C}(S_{\mathcal{Q}})$. For $a \in \mathcal{C}(S_{Sp})$ and $c \in \mathcal{C}(S_{\mathcal{Q}})$, we define

$$a_+(c) = \text{Surger}(|\hat{\imath}|(a,c)a, c), \text{ and}$$
$$a_-(c) = \text{Surger}(c, |\hat{\imath}|(a,c)a).$$

Lemma 6.4. *For $a \in \mathcal{C}(S_{Sp})$ and $c \in \mathcal{C}(S_{\mathcal{Q}})$, we have that $a_+(c)$ is a (connected) simple closed curve surrounding an even number of marked points. If $\tilde{a}$, $\tilde{c}$, and $\widetilde{a_+(c)}$ are connected components of the preimages of a, c, and $a_+(c)$ in Σ_g^1, then, up to choosing compatible orientations on $\tilde{c}$ and $\widetilde{a_+(c)}$, we have*

$$\overline{t}_a([\tilde{c}]) = \left[\widetilde{a_+(c)}\right].$$

Similarly, $a_-(c)$ surrounds an even number of marked points and $\overline{t}_a^{-1}([\tilde{c}]) = \left[\widetilde{a_-(c)}\right]$.

Proof. We only treat the case of $a_+(c)$ with the other case being completely analogous. We begin with the first statement. The geometric intersection number $i(a, c)$ is equal to 0, 2, or 4; this is because c_{ij} is the boundary of a regular neighborhood of the straight line segment connecting p_i to p_j, and a straight line can intersect a convex curve in 0, 1, or 2 points. We treat each of the three cases in turn.

If $i(a, c) = 0$, then $|\hat{\imath}(a, c)| = 0$. Thus, $a_+(c)$ is equal to c, which is a simple closed curve.

If $i(a, c) = 2$, then we claim that $|\hat{\imath}|(a, c) = 1$. Indeed, the arc of a crossing through c necessarily separates the two marked points inside c from each other, creating two bigons, each containing one marked point. The preimage of one bigon in Σ_g^1 is a square whose four corners are the four intersection points of the prieimages of a and c. We know that the hyperelliptic involution ι interchanges the two lifts of each curve and that ι rotates the square by angle π. Our claim follows. It thus remains to check that Surger(a, c) is a simple closed curve, which is immediate from Figure 7.

If $i(a, c) = 4$, then we claim that $|\hat{\imath}|(a, c)$ is equal to either 0 or 2, depending on whether the arcs of c divide the marked points inside a into two sets of even cardinality or odd cardinality, respectively. The curve a divides the convex region bounded by c into three connected components: one square and two bigons, each with one marked point. Consider the union of the square and one bigon. The preimage in Σ_g^1 is a rectangle made up of three squares; there is one central square (the preimage of the bigon) and two other squares with edges glued to the left and right edges of the central square. Since each intersection point in

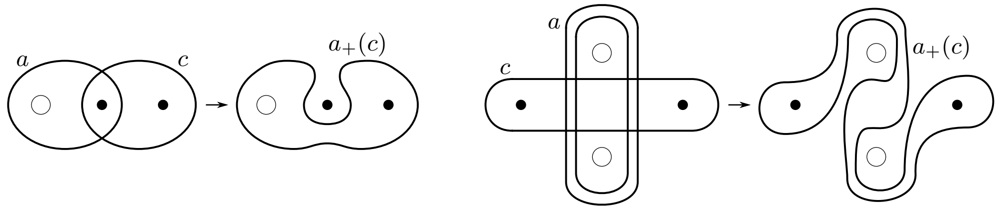

Figure 7: *Left: If $i(a,c)=2$, then* Surger(a,c) *is a simple closed curve. Right: If $i(a,c)=4$, then* Surger$(2a,c)$ *is a simple closed curve. In both figures, the small unfilled circles contain an unspecified (but nonzero) number of marked points.*

D_{2g+1} lifts to two intersection points in Σ_g^1, and since we already see 8 intersection points on the boundary of this rectangle, we conclude that this picture contains all of the intersection points of preimages of a and c. Also, by construction the horizontal sides of the rectangle belong to preimages of c. The involution ι acts on this rectangle, rotating it by π though the center. We also know that ι interchanges the two preimages of each of a and c. Therefore, it suffices to count the intersections of the bottom of the rectangle with the vertical sides of the rectangle belonging to a single component of the preimage of a. Again because ι exchanges the two components of the preimage of a, two of the vertical segments belong to one component, and two to the other. Thus, if we choose one component of the preimage of a, it intersects the bottom edge of the rectangle in precisely two points. It immediately follows that $|\hat{\imath}(a,c)|$ is equal to either 0 or 2, as claimed. By the claim, it suffices to check that Surger$(2a,c)$ is a simple closed curve, which is also immediate from Figure 7.

We now address the second statement of the lemma. The preimage in Σ_g^1 of the configuration $|\hat{\imath}|(a,c)a \cup c$ is a symmetric configuration (that is, a subset preserved by ι) with $|\hat{\imath}|(a,c)$ copies of each preimage of a and both preimages of c. We orient these so all preimages of a represent the same element of $H_1(\Sigma_g^1;\mathbb{Z})$. We do the same for c; there are two choices, and we use the one that is consistent with the surgery in Figure 6. When we perform surgery on this configuration, we therefore obtain a symmetric representative of the homology class

$$2[\tilde{c}] + 2|\hat{\imath}|(a,c)[\tilde{a}].$$

This symmetric representative is the preimage of $a_+(c)$ and so the first statement of the lemma implies that this representative has exactly two connected components that are interchanged by the hyperelliptic involution. It follows that each component, in particular $\widetilde{a_+(c)}$, represents

$$\tau_{[\tilde{a}]}([\tilde{c}]) = [\tilde{c}] + |\hat{\imath}|(a,c)[\tilde{a}].$$

But (up to sign) this is equal to $\overline{t}_a([\tilde{c}])$, and the lemma is proven. $\square$

We can now define our action of $S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ on $S_{\mathcal{Q}}$. Consider $t_a \in S_{\mathrm{Sp}}$ and $s_c \in S_{\mathcal{Q}}$. By Lemma 6.4, $a_+(c)$ is a simple closed curve surrounding an even number of marked points and so there is a well-defined transvection $t_{a_+(c)}$ (similarly for $a_-(c)$). We set

$$t_a \cdot s_c = \overline{s}_{a_+(c)} \in \mathcal{Q}_g \text{ and}$$
$$t_a^{-1} \cdot s_c = \overline{s}_{a_-(c)} \in \mathcal{Q}_g.$$

These are both well-defined elements of $\mathcal{Q}_g$ since $s_{a_\pm(c)}$ only depends on the homotopy class of $a_\pm(c)$, and we already said that the latter is a well-defined simple closed curve. It remains to verify the naturality property (1), which is a consequence of the following.

Lemma 6.5. *For any $t_a \in S_{\mathrm{Sp}}$ and $s_c \in S_{\mathcal{Q}}$ and $\epsilon \in \{-1, 1\}$, we have*

$$\pi\left(t_a^\epsilon \cdot s_c\right) = \overline{t}_a^\epsilon \pi(\overline{s}_c)\overline{t}_a^{-\epsilon}.$$

Proof. To simplify notation, we will treat the case $\epsilon = 1$; the other case is essentially the same. Let $\widetilde{a_+(c)}$ be one component in Σ_g^1 of the preimage of $a_+(c)$. We have that

$$\pi(t_a \cdot s_c) = \pi(\,\overline{s}_{a_+(c)}\,) = \tau^2_{\left[\widetilde{a_+(c)}\right]} = \tau^2_{\overline{t}_a(\tilde{c})} = \overline{t}_a \tau^2_{[\tilde{c}]} \overline{t}_a^{-1} = \overline{t}_a \pi(\overline{s}_c)\overline{t}_a^{-1},$$

as desired. In turn, the five equalities are given by the definition of our action, Lemma 6.3, Lemma 6.4, Equation (2), and Lemma 6.3. ☐

6.4 Well-definedness with respect to $\mathcal{Q}_g$ relations

Recall that $\hat{f}_t : F(S_{\mathcal{Q}}) \to \mathcal{Q}_g$ is the unique homomorphism satisfying $\hat{f}_t(s) = t \cdot s$ for all $s \in S_{\mathcal{Q}}$. In this section we prove Lemma 6.1, which states that for all $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $r \in R_{\mathcal{Q}}$, we have $\hat{f}_t(r) = 1$.

We introduce the following terminology, which will also be used in Section 6.5. Let d be an essential simple closed curve in D_{2g+1}. An element of $\mathcal{Q}_g$ is said to be *reducible along d* if it is the image of an element of PB_{2g+1} that preserves the isotopy class of d.

Lemma 6.6. *If $\eta \in \mathcal{Q}_g$ is reducible and $\pi(\eta) = 1$, then $\eta = 1$.*

Proof. The condition $\pi(\eta) = 1$ implies that η lies in the image of $\mathcal{BI}_{2g+1}$ in $\mathcal{Q}_g$. Recall that we are assuming that $\mathcal{BI}_{2h+1} = \Theta_{2h+1}$ for $h < g$, so Theorem 2.2 implies that reducible elements of $\mathcal{BI}_{2g+1}$ lie in Θ_{2g+1} (and hence map to 1 in $\mathcal{Q}_g$). Since η is reducible, we conclude that $\eta = 1$. ☐

Lemma 6.7. *Let $r \in F(S_{\mathcal{Q}})$ be a relator in $\mathcal{Q}_g$, let $t_a \in S_{\mathrm{Sp}}$, and let $\epsilon = \pm 1$. Then $\pi(\hat{f}_{t_a^\epsilon}(r)) = 1$.*

Proof. Write $r = s_{c_{i_1 j_1}}^{\epsilon_1} \cdots s_{c_{i_n j_n}}^{\epsilon_n}$, with $\epsilon_i = \pm 1$. By Lemma 6.5, we have

$$\pi(\hat{f}_{t_a^\epsilon}(r)) = \pi(t_a^\epsilon \cdot s_{c_{i_1 j_1}}^{\epsilon_1}) \cdots \pi(t_a^\epsilon \cdot s_{c_{i_n j_n}}^{\epsilon_n}) = \overline{t}_a^\epsilon \pi(\overline{r})\overline{t}_a^{-\epsilon} = 1,$$

as desired. ☐

Lemma 6.8. *Let $t_a \in S_{\mathrm{Sp}}$, let $\epsilon = \pm 1$, and let $r \in F(S_{\mathcal{Q}})$ be a relator in $\mathcal{Q}_g$. Suppose that there is an essential simple closed curve d in D_{2g+1} disjoint from a and from each curve c of $\mathcal{C}(S_{\mathcal{Q}})$ such that $s_c^{\pm 1}$ appears in r. Then $\hat{f}_{t_a^\epsilon}(r) = 1$.*

Proof. Write $r = s_{c_{i_1 j_1}}^{\epsilon_1} \cdots s_{c_{i_n j_n}}^{\epsilon_n}$, with $\epsilon_i = \pm 1$. By hypothesis, each $c_{i_k j_k}$ is disjoint from d. We have

$$\hat{f}_{t_a^\epsilon}(r) = (t_a^\epsilon \cdot s_{c_{i_1 j_1}}^{\epsilon_1}) \cdots (t_a^\epsilon \cdot s_{c_{i_n j_n}}^{\epsilon_n}).$$

By the definition of $t_a^\epsilon \cdot s_{c_{i_kj_k}}^{\epsilon_k}$ and the fact that a and each $c_{i_kj_k}$ is disjoint from d, it follows that $t_a^\epsilon \cdot s_{c_{i_kj_k}}$ is reducible along d for all $1 \leq k \leq n$, so $\hat{f}_{t_a^\epsilon}(r)$ is reducible along d. By Lemma 6.7, $\pi(\hat{f}_{t_a^\epsilon}(r)) = 1$. Lemma 6.6 thus implies that $\hat{f}_{t_a^\epsilon}(r) = 1$. ☐

Finally, in order to prove Lemma 6.1 we need the following two lemmas, which give generating sets for two different kinds of subgroups of PB_{2g+1}. The first follows easily from the fact that any inclusion $\mathrm{D}_n \to \mathrm{D}_{2g+1}$ induces an inclusion on the level of mapping class groups [17, Theorem 3.18].

Lemma 6.9. *Let Δ be a convex subdisk of D_{2g+1} containing n marked points. The subgroup of PB_{2g+1} consisting of elements with representatives supported in Δ is isomorphic to PB_n, and is generated by the Dehn twists $T_{c_{ij}}$ with $p_i, p_j \in \Delta$.*

Lemma 6.10. *Let $1 \leq i < j \leq 2g+1$. The stabilizer in PB_{2g+1} of the curve c_{ij} is generated by the Dehn twists about curves in the set*

$$\{c_{ij}\} \cup \{c_{k\ell} \mid k, \ell \notin \{i,j\}\} \cup \{c_{ijk} \mid k \notin \{i,j\}\}.$$

Proof. Let $(\mathrm{PB}_{2g+1})_{c_{ij}}$ denote the stabilizer in PB_{2g+1} of c_{ij}, and let γ_{ij} denote the straight line segment connecting p_i to p_j. Any element of the group $(\mathrm{PB}_{2g+1})_{c_{ij}}$ has a representative that preserves γ_{ij}. Any such homeomorphism descends to a homeomorphism of the disk with $2g$ marked points obtained from D_{2g+1} by collapsing γ_{ij} to a single marked point. This procedure gives rise to a short exact sequence:

$$1 \to \langle T_{c_{ij}} \rangle \to (\mathrm{PB}_{2g+1})_{c_{ij}} \to \mathrm{PB}_{2g} \to 1.$$

The proof is the same as that of Proposition 3.20 in [17].

The Dehn twist $T_{c_{ij}}$ lies in the set of generators given in the statement of the lemma, and the Dehn twists about the other curves in the statement of the lemma map to the Artin generators for PB_{2g}. The lemma follows. ☐

Proof of Lemma 6.1. The proof of this lemma will be broken into two steps. For the first, let $R_{\mathrm{PB}} \subset R_{\mathcal{Q}}$ be the subset consisting of the disjointness, triangle, and crossing relations. As was observed in Section 6.2.3, we have $\mathrm{PB}_{2g+1} \cong \langle S_{\mathcal{Q}} \mid R_{\mathrm{PB}} \rangle$.

Step 1. For $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $r \in R_{\mathrm{PB}}$, we have $\hat{f}_t(r) = 1$.

Write $t = t_a^\epsilon$ for $\epsilon = \pm 1$. By Lemma 6.8, it suffices to find an essential simple closed curve d in D_{2g+1} disjoint from a and each curve of $\mathcal{C}(S_{\mathcal{Q}})$ that appears in r.

Denote by Δ_r the convex hull of curves of $\mathcal{C}(S_{\mathcal{Q}})$ that appear in r. Examining the relations in R_{PB}, we see that Δ_r contains at most 4 marked points, and hence there are at least 3 marked points outside of Δ_r.

The intersection number of any element of $\mathcal{C}(S_{Sp})$ with any element of $\mathcal{C}(S_{\mathcal{Q}})$ is at most 4. It follows that the intersection of a with the closure of the exterior of Δ_r is a union of at most two arcs. These two arcs partition the marked points outside of Δ_r into at most three sets. Thus, it must either be the case that there is a pair of marked points that can

be connected by an arc α disjoint from $a \cup \Delta_r$, or it must be the case that the convex hull of of $a \cup \Delta_r$ contains at least one marked point in its exterior. In the first case, we can take d to be the boundary of a regular neighborhood of α and in the second case we can take d to be the boundary of the convex hull of $a \cup \Delta_r$.

Step 2. For $t \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$ and $r \in R_{\mathcal{Q}}$ an odd twist relator, $\hat{f}_t(r) = 1$.

Again write $t = t_a^\epsilon$ for $\epsilon = \pm 1$. Consider $B \subset \{1, \ldots, 2g+1\}$ with $3 \leq |B| \leq 2g+1$ and $|B|$ odd. There is an odd twist relator r_B corresponding to B and its image under the map $F(S_{\mathcal{Q}}) \to \mathrm{PB}_{2g+1}$ is $T_{c_B}^2$. We need to show $\hat{f}_t(r_B) = 1$.

It follows from Step 1 that $\hat{f}_t$ factors through a homomorphism $\mathrm{PB}_{2g+1} \to \mathcal{Q}_g$. This means that if two elements of $F(S_{\mathcal{Q}})$ have the same image in PB_{2g+1}, then they have the same image under $\hat{f}_t$. Thus, we have the luxury of rewriting the odd twist relator r_B as an equivalent relator r'_B that satisfies the hypotheses of Lemma 6.8.

To this end, we make the following observation: for any $a \in \mathcal{C}(S_{Sp})$ and any convex simple closed curve d in D_{2g+1}, we have $i(a, d) \leq 4$. (An example of a curve a that does not satisfy this property is $a = c_{1246}$; we chose our presentation carefully to avoid such curves.) In particular, $i(c_B, a) \leq 4$. Thus $c_B \cup a$ separates D_{2g+1} into at most 6 regions. The region containing the boundary of D_{2g+1} is not convex, but the other regions are (homotopic to) convex ones. Since $g \geq 3$, there are at least 7 marked points, and so one component must contain at least two marked points.

First suppose the exterior region contains at least one marked point, say p_k. Let $A_k = \{1, \ldots, \hat{k}, \ldots, 2g+1\}$. By Lemma 6.9, we can write $T_{c_B}^2$ as a product of Dehn twists (and inverse Dehn twists) about the c_{ij} where $i, j \neq k$. More to the point, there is a product r'_B of $s_{c_{ij}}^{\pm 1} \in F(S_{\mathcal{Q}})$ with $i, j \neq k$ whose image in PB_{2g+1} is equal to $T_{c_B}^2$. Since $T_{c_B}^2$ lies in Θ_{2g+1}, we have that r'_B is a relator for $\mathcal{Q}_g$. Since a and each c_{ij} with $i, j \neq k$ is disjoint from c_{A_k}, Lemma 6.8 gives that $\hat{f}(r'_B)$, hence $\hat{f}(r_B)$, is equal to 1, as desired.

Finally suppose that one of the convex components contains two marked points p_i and p_j. In this case the curve c_{ij} is disjoint from both a and c_B. By Lemma 6.10, we can write $T_{c_B}^2$ in terms of the Dehn twists about the curves in the set

$$\mathcal{C}_{ij} = \{c_{ij}\} \cup \{c_{k\ell} \mid k, \ell \notin \{i, j\}\} \cup \{c_{ijk} \mid k \notin \{i, j\}\}.$$

We now treat two subcases. First suppose neither i nor j lies in B. By Lemma 6.9, we can write $T_{c_B}^2$ as a product of $T_{c_{k\ell}}^{\pm 1}$ with $i, j \notin \{k, \ell\}$. Each such $c_{k\ell}$ is disjoint from c_{ij}, as is a. By the same argument as the last case, $\hat{f}(r_B) = 1$.

The second subcase is that $i, j \in B$. In this case we rewrite r_B as a product r'_B of arbitrary elements of $\mathcal{C}_{ij}$. We wish to show that $\hat{f}_t(r'_B)$ is reducible along c_{ij}, for then it immediately follows from Lemmas 6.7 and 6.8 that $\hat{f}_t(r'_B) = 1$. We know from the definitions that $t_a^\epsilon \cdot c_{k\ell}$ is reducible along c_{ij} for $c_{k\ell} \in \mathcal{C}_{ij}$. It remains to show that $t_a^\epsilon \cdot c_{ijk}$ is reducible along c_{ij} when $c_{ijk} \in \mathcal{C}_{ij}$. But for any such choice of $d = c_{ijk}$ and any choice of $a \in \mathcal{C}(S_{Sp})$, the triple (c_{ij}, a, d) satisfies the hypotheses of Lemma 6.11 below, so we are done. ☐

Lemma 6.11. *Assume that $\hat{f}_t$ descends to a well-defined homomorphism $\mathrm{PB}_{2g+1} \to \mathcal{Q}_g$ (that is, assume Step 1 of the proof of Lemma 6.1). Fix some $c = c_{ij} \in \mathcal{C}(S_{\mathcal{Q}})$. Let*

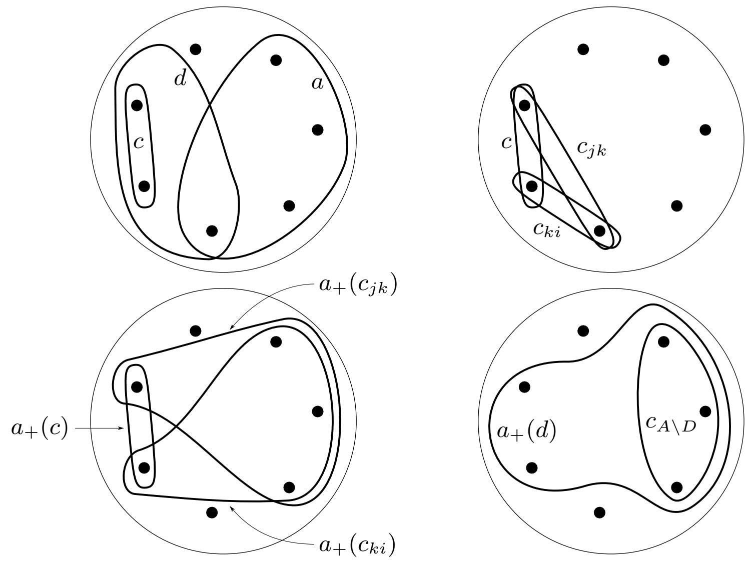

Figure 8: *The curves curves used in the $|A \cap D| = 1$ case of Lemma 6.11.*

$a = c_A \in \mathcal{C}(S_{Sp})$ with $i(a, c) = 0$ and let $d = c_D \in \{c_{ijk} \mid k \notin \{i, j\}\}$. Then

$$t_a^{\pm 1} \cdot s_d = \begin{cases} \overline{s}_d & |A \cap D| = 3 \\ \overline{s}_{a_\pm(d)}\ \overline{s}_{c_{A\cup D}} & |A \cap D| = 2 \\ \overline{s}_{a_\pm(d)}\ \overline{s}_{c_{A\setminus D}} & |A \cap D| = 1 \\ \overline{s}_d & |A \cap D| = 0 \text{ and } i(a,d) = 0 \\ \overline{s}_{a_\pm(d)} & |A \cap D| = 0 \text{ and } i(a,d) = 4 \end{cases}$$

In particular, $t_a^{\pm 1} \cdot s_d$ is reducible along c.

Proof. When $|A \cap D|$ is equal to 3, we have $i(a, d) = 0$, and the action is trivial by definition. For the next case, say the elements of D are $k < i < j$. By the lantern relation [17, Proposition 5.1], we can factor the Dehn twist T_d as $T_d = T_{c_{ij}} T_{c_{jk}} T_{c_{ki}}$. Since we are assuming that $\hat{f}_t$ descends to a homomorphism $\mathrm{PB}_{2g+1} \to \mathcal{Q}_g$, we have that

$$t_a \cdot s_d = (t_a \cdot s_{c_{ij}})(t_a \cdot s_{c_{jk}})(t_a \cdot s_{c_{ki}}).$$

By the lantern relation again, the latter is equal to $\overline{s}_{a_\pm(d)} \overline{s}_{c_{A\cup D}}$, as desired; see Figure 8. The third case is similar. The fourth case is trivial. The last case, where $|A \cap D| = 0$, has two subcases: either c_{jk} and c_{ki} each separate A into two sets of even cardinality, or they each separate A into two sets of odd cardinality. In the first subcase $\hat{\imath}(a, c_{ij}) = \hat{\imath}(a, c_{jk}) = \hat{\imath}(a, c_{ki}) = 0$ and in the second subcase $\hat{\imath}(a, c_{ij}) = 0$ and $\hat{\imath}(a, c_{jk}) = \hat{\imath}(a, c_{ki}) = 2$ (cf. the proof of Lemma 6.4). In the first subcase there is nothing to check. The second subcase is handled by an application of the lantern relation, similar to the $|A \cap D| = 2$ case above. $\square$

6.5 Well-definedness with respect to $\mathrm{Sp}_{2g}(\mathbb{Z})$ relations

In this section we prove Lemma 6.2, which states that, for any $r \in F(S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1})$ such that either $r \in R_{\mathrm{Sp}}$ or $r = ss^{-1}$ with s a generator, we have that r acts trivially on $\mathcal{Q}_g$.

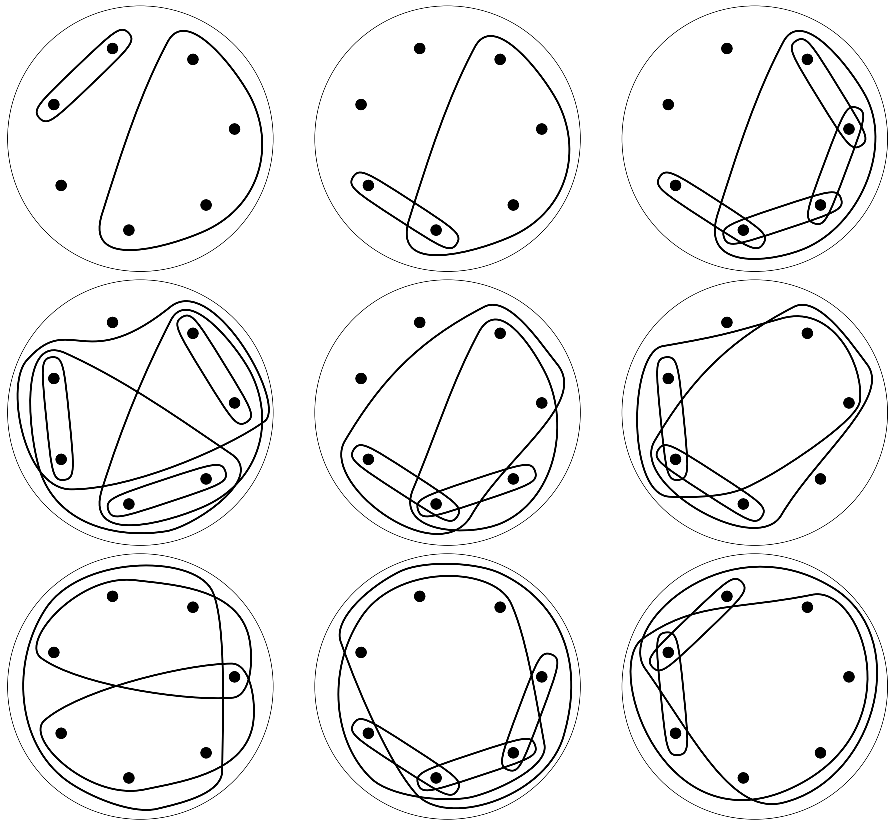

Figure 9: *Configurations of curves from $\mathcal{C}(S_{Sp})$ arising in the relators for* $\mathrm{Sp}_{2g}(\mathbb{Z})$*. From the top left moving right: a disjointness relation, a braid relation, the chain relation, the lantern relation, and auxiliary relations (i), (ii), (vii), (viii), and (ix).*

There are two steps. First we will observe that each relator r as above satisfies the reducibility criterion stated below, and then we will show that any such relator satisfying the reducibility criterion also satisfies the conclusion of the Lemma 6.2.

Reducibility criterion. We say that a relator r for $\mathrm{Sp}_{2g}(\mathbb{Z})$ satisfies the *reducibility condition* if the union of the curves of $\mathcal{C}(S_{\mathrm{Sp}})$ appearing in r together with any $c \in \mathcal{C}(S_{\mathcal{Q}})$ is disjoint from some straight line segment in D_{2g+1} connecting two marked points or a marked point to the boundary.

It is straightforward to check that each relator $r \in R_{\mathrm{Sp}}$ or $r = ss^{-1}$ with $s \in S_{\mathrm{Sp}}$ satisfies the reducibility criterion. See Figure 9 for a representative collection of relators. This is one place where our specific choice of presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$ comes into play. If we were to use Wajnryb's presentation without our auxiliary generators, some of the relations (like the lantern relation) would fill the entire disk.

It remains to verify that any relator $r \in R_{\mathrm{Sp}}$ or $r = ss^{-1}$ satisfying the reducibility criterion necessarily satisfies the conclusion of Lemma 6.2. To this end, let $c = c_{ij} \in \mathcal{C}(S_{\mathcal{Q}})$. It suffices to show that $r \cdot c = c$. Write $r = t_1 \cdots t_n$ where $t_i \in S_{\mathrm{Sp}} \cup S_{\mathrm{Sp}}^{-1}$.

There are two cases. First suppose there is a straight line segment that joins a marked point p_k to the boundary of D_{2g+1} and that is disjoint from c and each curve of $\mathcal{C}(S_{\mathrm{Sp}})$

that appears in r. Let d denote the convex simple closed curve surrounding all the marked points but p_k.

By the definition of our action, we have that $t_n \cdot s_c$ is reducible along d and moreover $t_n \cdot s_c$ is equal to $\rho(b_n)$, where $b_n \in \mathrm{PB}_{2g+1}$ has a representative homeomorphism supported in the interior of d. By Lemma 6.9, we can write $t_n \cdot s_c$ as the image in $\mathcal{Q}_g$ of a product of Dehn twists (and inverse Dehn twists) about curves that surround two marked points and are disjoint from d. It then follows from the definition of the action that $t_{n-1} \cdot (t_n \cdot s_c)$ is reducible along d and is equal to $\rho(b_{n-1})$, where b_{n-1} is represented by a homeomorphism supported in the interior of d. Continuing inductively, we deduce that $r \cdot s_c$ is reducible along d. Since s_c is also reducible along d, we have that $(r \cdot s_c)s_c^{-1}$ is reducible along d. By Lemma 6.5, $\pi(r \cdot s_c)\pi(s_c)^{-1} = 1$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$. Thus by Lemma 6.7, $(r \cdot s_c)s_c^{-1}$ is equal to the identity in $\mathcal{Q}_g$, as desired.

Now suppose there is a straight line segment connecting two marked points in D_{2g+1} and disjoint from c and each curve of $\mathcal{C}(S_{\mathrm{Sp}})$ that appears in r. Let $d = c_{k\ell}$ denote the boundary of a regular neighborhood of this line segment. The argument is similar to the previous case. The only difference is that when we factor the preimage of $t_n \cdot s_c$ in PB_{2g+1}, we must use Dehn twists about curves that surround two or three marked points and are disjoint from d. However, we can use the same argument, applying Lemma 6.11 as necessary. This completes the proof.

6.6 Completing the proof

At this point, we have established the existence of a specific action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{Q}_g$. It remains to check that this action has all three properties stipulated by Proposition 3.4.

We already mentioned that property (1), namely, that $\pi(Z \cdot \eta) = Z\pi(\eta)Z^{-1}$ for $Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\eta \in \mathcal{Q}_g$, follows directly from Lemma 6.5.

Property (2) asserts that $\widehat{\pi}(\nu) \cdot \eta = \nu\eta\nu^{-1}$ for $\nu \in \widehat{\mathcal{Q}}_g$ and $\eta \in \mathcal{Q}_g$. For $a_i = c_{i,i+1}$, the half-twist H_{a_i} in B_{2g+1} is the unique square root of T_{a_i}. The half-twists about $a_1, \ldots, a_{2g}$ are the usual generators for the braid group B_{2g+1}. The element H_{a_i} maps to $\overline{t}_{a_i} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, so to prove property (2) it is enough to show that

$$t_{a_i}^{\epsilon} \cdot s_{c_{jk}} = \widehat{\rho}(H_{a_i}^{\epsilon})\, \overline{s}_{c_{jk}}\, \widehat{\rho}(H_{a_i}^{-\epsilon})$$

for all choices of i, j, k. Since $\overline{s}_{c_{jk}} = \rho(T_{c_{jk}})$, the right hand side is equal to $\overline{s}_{H_{a_i}^{\epsilon}(c_{jk})} = \rho(T_{H_{a_i}^{\epsilon}(c_{jk})})$, and so it remains to show that $(a_i)_\epsilon(c_{jk}) = H_{a_i}^{\epsilon}(c_{jk})$, where $(a_i)_\epsilon$ is interpreted as either $(a_i)_+$ or $(a_i)_-$. For any choices of i, j, and k, we have $i(a_i, c_{jk})$ is either 0 or 2, and there is only one configuration in each case up to homeomorphisms of D_{2g+1}. In the case $i(a_i, c_{jk}) = 0$, we have $(a_i)_\pm(c_{jk}) = H_{a_i}^{\pm 1}(c_{jk}) = c_{jk}$. It remains to check the case $(i, j, k) = (1, 2, 3)$. It is easily checked that in this case $(a_1)_\epsilon(c_{23}) = H_{a_1}^{\epsilon}(c_{23})$.

We now turn to property (3) of Proposition 3.4, which states that the action of $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$ on $\mathcal{Q}_g$ preserves Ω_{23}; see the discussion before Proposition 3.4 for the definitions. We will use the fact that $(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\vec{v}_{23}}$ is generated by the set

$$\Xi = \{t_{a_2}, t_{u'}, t_{b_3}, t_{a_4}, t_{a_5}, \ldots, t_{a_{2g}}\}.$$

That Ξ generates is an immediate consequence of the split short exact sequence for the stabilizer in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of a primitive vector that is given in the proof of [37, Lemma 3.7].

Observe that $\Xi \subset S_{\mathrm{Sp}}$. Let Υ be the image in $\mathcal{Q}_g$ of the generating set for $(\mathrm{PB}_{2g+1})_{c_{23}}$ from Lemma 6.10. Each $y \in \Upsilon$ is reducible along c_{23}, and we have to show that for $x \in \Xi$ the element $x \cdot y$ is reducible along c_{23}. For the elements of Υ that lie in $S_{\mathcal{Q}}$, this is obvious from the description of our action in Section 6.3. For the others, it is an immediate consequence of Lemma 6.11. This completes the proof.

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